

# Constructing a quantum field theory from spacetime

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**Abstract** The paper shows deep connections between exotic smoothings of a small  $\mathbb{R}^4$  (the spacetime), the leaf space of codimension-1 foliations (related to noncommutative algebras) and quantization. At first we relate a small exotic  $\mathbb{R}^4$  to codimension-1 foliations of the 3-sphere unique up to foliated cobordisms and characterized by the real-valued Godbillon-Vey invariant. Special care is taken for the integer case which is related to flat  $PSL(2, \mathbb{R})$ -bundles. Then we discuss the leaf space of the foliation using noncommutative geometry. This leaf space contains the hyperfinite  $III_1$  factor of Araki and Woods important for quantum field theory (QFT) and the  $I_\infty$  factor. Using Tomita's modular theory, one obtains a relation to a factor  $II_\infty$  algebra given by the horocycle foliation of the unit tangent bundle of a surface  $S$  of genus  $g > 1$ . The relation to the exotic  $\mathbb{R}^4$  is used to construct the (classical) observable algebra as a Poisson algebra of functions over the character variety of representations of the fundamental group  $\pi_1(S)$  into the  $SL(2, \mathbb{C})$ . The Turaev-Drinfeld quantization (as deformation quantization) of this Poisson algebra is a (complex) skein algebra which is isomorphic to the hyperfinite factor  $II_1$  algebra determining the factor  $II_\infty = II_1 \otimes I_\infty$  algebra of the horocycle foliation. Therefore our geometrically motivated hyperfinite  $III_1$  factor algebra comes from the quantization of a Poisson algebra. Finally we discuss the states and operators to be knots and knot concordances, respectively.

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## 1 Introduction

The construction of quantum theories from classical theories, known as quantization, has a long and difficult history. It starts with the discovery of quantum mechanics in 1925 and the formalization of the quantization procedure by Dirac and von Neumann. The construction of a quantum theory from a given classical one is highly non-trivial and non-unique. But except for few examples, it is the only way which will be gone today. From a physical point of view, the world surround us is the result of an underlying quantum theory of its constituent parts. So, one would expect that we must understand the transition from the quantum to the classical world. But we had developed and tested successfully the classical theories like mechanics or electrodynamics. Therefore one tried to construct the quantum versions out of classical theories. In this paper we will go the other way to obtain a quantum field theory by geometrical methods and to show its equivalence to a quantization of a classical Poisson algebra.

The main technical tool will be the noncommutative geometry developed by Connes [24]. Then intractable space like the leaf space of a foliation can be described by non-commutative algebras. From the physical point of view, we have now an interpretation of noncommutative algebras (used in quantum theory) in a geometrical context. So, we need only an idea for the suitable geometric structure. For that purpose one formally considers the path integral over spacetime geometries. In the evaluation of this integral, one has to include the possibility of different smoothness structures for spacetime [45, 7]. Brans [17, 16, 15] was the first who considered exotic smoothness also on open smooth 4-manifolds as a possibility for space-time. He conjectured that exotic smoothness induces an additional gravitational field (*Brans conjecture*). The conjecture was established by Asselmeyer [6] in the compact case and by Śladowski [51] in the non-compact case. Śladowski [50, 48, 49] discussed the influence of differential structures on the algebra  $C(M)$  of functions over the manifold  $M$  with methods known as non-commutative geometry. Especially in [48, 49] he stated a remarkable connection between the spectra of differential operators and differential structures. But there is a big problem which prevents progress in the understanding of exotic smoothness especially for the  $\mathbb{R}^4$ : there is no known explicit coordinate representation. As the result no exotic smooth function on any such  $\mathbb{R}^4$  is known even though there exist families of infinite continuum many different non diffeomorphic smooth  $\mathbb{R}^4$ . This is also a strong limitation for the applicability to physics of non-standard open 4-smoothness. Bizaca [11] was able to construct an infinite coordinate patch by using Casson handles. But it still seems hopeless to extract physical information from that approach.

This situation is not satisfactory but we found a possible solution. The solution is a careful analysis of the small exotic  $\mathbb{R}^4$  by using foliation theory (see next section) to derive a relation between exotic smoothness and codimension-1 foliations in section 3 (see Theorem 5). By using noncommutative geometry, this approach is able to produce a von Neumann algebra via the leaf space of the foliation which can be interpreted as the observable algebra of some QFT (see [37]). Fortunately, our approach to exotic smoothness is strongly connected with a codimension-1 foliation of type *III*, i.e. the leaf space is a factor *III*<sub>1</sub> von Neumann algebra. Especially this algebra is the preferred algebra in the local algebra approach to QFT [37, 14]. Recently, this factor *III* case was also discussed in connection with quantum gravity (via the spectral triple of Connes) [10].

In the next two sections we will give an overview about foliation theory, its operator-theoretical description and the relation to exotic smoothness. Both sections are rather

technical with a strong overlap to our previous paper [9]. In section 4 we turn to the quantization procedure as related to nonstandard smoothings of  $\mathbb{R}^4$ . Based on the dictionary between operator algebra and foliations one has the corresponding relation of small exotic  $\mathbb{R}^4$ 's and operator algebras. This is a noncommutative  $C^*$  algebra which can be seen as the algebra of quantum observables of some theory.

- First, in subsection 2.5 we recognized the algebra as the hyperfinite factor  $III_1$  von Neumann algebra. From Tomita-Takesaki theory it follows that any factor  $III$  algebra  $M$  decomposes as a crossed product into  $M = N \rtimes_{\theta} \mathbb{R}_+^*$  where  $N$  is a factor  $II_{\infty}$ . Via Connes procedure one can relate the factor  $III$  foliation to a factor  $II$  foliation. Then we obtain a foliation of the horocycle flow on the unit tangent bundle over some genus  $g$  surface which determines the factor  $II_{\infty}$ . This foliation is in fact determined by the horocycles which are closed circles.
- Next we are looking for a classical algebraic structure which would give the above mentioned noncommutative algebra of observables as a result of quantization. The classical structure is recovered by the idempotent of the  $C^*$  algebra and has the structure of a Poisson algebra. The idempotents were already constructed in subsection 2.4 as closed curves in the leaf of the foliation of  $S^3$ . As noted by Turaev [60], closed curves in a surface induce a Poisson algebra: Given a surface  $S$  let  $X(S, G)$  be the space of flat connections of  $G = SL(2, \mathbb{C})$  bundles on  $S$ ; this space carries a Poisson structure as is shown in subsection 4.2. The complex functions on  $X(S, SL(2, \mathbb{C}))$  can be considered as the algebra of classical observables forming the Poisson algebra  $(X(S, SL(2, \mathbb{C})), \{, \})$ .
- Next in the subsection 4.3 we find a quantization procedure of the above Poisson algebra which is the Drinfeld-Turaev deformation quantization. It is shown that the result of this quantization is the skein algebra  $(K_t(S), [, ])$  for the deformation parameter  $t = \exp(h/4)$  ( $t = -1$  corresponds to the commutative Poisson structure on  $(X(S, SL(2, \mathbb{C})), \{, \})$ ).
- This skein algebra is directly related to the factor  $III_1$  von Neumann algebra derived from the foliation of  $S^3$ . In fact the skein algebra is constructed in subsection 4.4 as the factor  $II_1$  algebra Morita equivalent to the factor  $II_{\infty}$  which in turn determines the factor  $III_1$  of the foliation.

Finally in section 5 we discuss the states of the algebra and the operators between states. Here, we present only the ideas: the states are knots represented by holonomies along a flat connection. Then an operator between two states is a knot concordance (a kind of knot cobordism). The whole approach is similar to the holonomy flux algebra of Loop quantum gravity (see [43]). We will discuss this interesting relation in our forthcoming work.

## 2 Preliminaries: Foliations and Operator algebras

In this section we will consider a foliation  $(M, F)$  of a manifold  $M$ , i.e. an integrable subbundle  $F \subset TM$  of the tangent bundle  $TM$ . The leaves  $L$  of the foliation  $(M, F)$  are the maximal connected submanifolds  $L \subset M$  with  $T_x L = F_x \forall x \in L$ . We denote with  $M/F$  the set of leaves or the leaf space. Now one can associate to the leaf space  $M/F$  a  $C^*$  algebra  $C(M, F)$  by using the smooth holonomy groupoid  $G$  of the foliation (see Connes [23]). For a codimension-1 foliation of a 3-manifold  $M$  there is the Godbillon-Vey invariant [31] as element of  $H^3(M, \mathbb{R})$ . As example we consider the construction

of a codimension-1 foliation of the 3-sphere by Thurston [57] which will be used extensively in the paper. This foliation has a non-trivial Godbillon-Vey invariant where every element of  $H^3(S^3, \mathbb{R})$  is represented by a cobordism class of foliations. Hurder and Katok [39] showed that the  $C^*$ -algebra of a foliation with non-trivial Godbillon-Vey invariant contains a factor *III* subalgebra. In the following we will construct this  $C^*$ -algebra and discuss the factor *III* case.

## 2.1 Definition of Foliations and foliated cobordisms

A codimension  $k$  foliation<sup>1</sup> of an  $n$ -manifold  $M^n$  (see the nice overview article [42]) is a geometric structure which is formally defined by an atlas  $\{\phi_i : U_i \rightarrow M^n\}$ , with  $U_i \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ , such that the transition functions have the form

$$\phi_{ij}(x, y) = (f(x, y), g(y)), \quad [x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k] \quad .$$

Intuitively, a foliation is a pattern of  $(n - k)$ -dimensional stripes - i.e., submanifolds - on  $M^n$ , called the leaves of the foliation, which are locally well-behaved. The tangent space to the leaves of a foliation  $\mathcal{F}$  forms a vector bundle over  $M^n$ , denoted  $T\mathcal{F}$ . The complementary bundle  $\nu\mathcal{F} = TM^n/T\mathcal{F}$  is the normal bundle of  $\mathcal{F}$ . Such foliations are called regular in contrast to singular foliations or Haefliger structures. For the important case of a codimension-1 foliation we need an overall non-vanishing vector field or its dual, an one-form  $\omega$ . This one-form defines a foliation iff it is integrable, i.e.

$$d\omega \wedge \omega = 0$$

and the leaves are the solutions of the equation  $\omega = \text{const.}$

Now we will discuss an important equivalence relation between foliations, cobordant foliations. Let  $M_0$  and  $M_1$  be two closed, oriented  $m$ -manifolds with codimension- $q$  foliations. Then these foliated manifolds are said to be *foliated cobordant* if there is a compact, oriented  $(m + 1)$ -manifold with boundary  $\partial W = M_0 \sqcup \overline{M}_1$  and with a codimension- $q$  foliation transverse to the boundary and inducing the given foliation there. The resulting foliated cobordism classes form a group under disjoint union.

## 2.2 Non-cobordant foliations of $S^3$ detected by the Godbillon-Vey class

In [57], Thurston constructed a foliation of the 3-sphere  $S^3$  depending on a polygon  $P$  in the hyperbolic plane  $\mathbb{H}^2$  so that two foliations are non-cobordant if the corresponding polygons have different areas. For later usage, we will present this construction now (see also the book [52] chapter VIII for the details).

Consider the hyperbolic plane  $\mathbb{H}^2$  and its unit tangent bundle  $T_1\mathbb{H}^2$ , i.e the tangent bundle  $T\mathbb{H}^2$  where every vector in the fiber has norm 1. Thus the bundle  $T_1\mathbb{H}^2$  is a  $S^1$ -bundle over  $\mathbb{H}^2$ . There is a foliation  $\mathcal{F}$  of  $T_1\mathbb{H}^2$  invariant under the isometries of  $\mathbb{H}^2$  which is induced by bundle structure and by a family of parallel geodesics on  $\mathbb{H}^2$ . The foliation  $\mathcal{F}$  is transverse to the fibers of  $T_1\mathbb{H}^2$ . Let  $P$  be any convex polygon in  $\mathbb{H}^2$ . We will construct a foliation  $\mathcal{F}_P$  of the three-sphere  $S^3$  depending on  $P$ . Let the

<sup>1</sup> In general, the differentiability of a foliation is very important. Here we consider the smooth case only.

sides of  $P$  be labeled  $s_1, \dots, s_k$  and let the angles have magnitudes  $\alpha_1, \dots, \alpha_k$ . Let  $Q$  be the closed region bounded by  $P \cup P'$ , where  $P'$  is the reflection of  $P$  through  $s_1$ . Let  $Q_\epsilon$ , be  $Q$  minus an open  $\epsilon$ -disk about each vertex. If  $\pi : T_1 \mathbb{H}^2 \rightarrow \mathbb{H}^2$  is the projection of the bundle  $T_1 \mathbb{H}^2$ , then  $\pi^{-1}(Q)$  is a solid torus  $Q \times S^1$  (with edges) with foliation  $\mathcal{F}_1$  induced from  $\mathcal{F}$ . For each  $i$ , there is a unique orientation-preserving isometry of  $\mathbb{H}^2$ , denoted  $I_i$ , which matches  $s_i$  point-for-point with its reflected image  $s'_i$ . We glue the cylinder  $\pi^{-1}(s_i \cap Q_\epsilon)$  to the cylinder  $\pi^{-1}(s'_i \cap Q_\epsilon)$  by the differential  $dI_i$  for each  $i > 1$ , to obtain a manifold  $M = (S^2 \setminus \{k \text{ punctures}\}) \times S^1$ , and a (glued) foliation  $\mathcal{F}_2$ , induced from  $\mathcal{F}_1$ . To get a complete  $S^3$ , we have to glue-in  $k$  solid tori for the  $k$   $S^1 \times \text{punctures}$ . Now we choose a linear foliation of the solid torus with slope  $\alpha_k/\pi$  (Reeb foliation). Finally we obtain a smooth codimension-1 foliation  $\mathcal{F}_P$  of the 3-sphere  $S^3$  depending on the polygon  $P$ .

Now we consider two codimension-1 foliations  $\mathcal{F}_1, \mathcal{F}_2$  depending on the convex polygons  $P_1$  and  $P_2$  in  $\mathbb{H}^2$ . As mentioned above, these foliations  $\mathcal{F}_1, \mathcal{F}_2$  are defined by two one-forms  $\omega_1$  and  $\omega_2$  with  $d\omega_a \wedge \omega_a = 0$  and  $a = 0, 1$ . Now we define the one-forms  $\theta_a$  as the solution of the equation

$$d\omega_a = -\theta_a \wedge \omega_a$$

and consider the closed 3-form

$$\Gamma_{\mathcal{F}_a} = \theta_a \wedge d\theta_a \quad (1)$$

associated to the foliation  $\mathcal{F}_a$ . As discovered by Godbillon and Vey [31],  $\Gamma_{\mathcal{F}}$  depends only on the foliation  $\mathcal{F}$  and not on the realization via  $\omega, \theta$ . Thus  $\Gamma_{\mathcal{F}}$ , the *Godbillon-Vey class*, is an invariant of the foliation. Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two cobordant foliations then  $\Gamma_{\mathcal{F}_1} = \Gamma_{\mathcal{F}_2}$ . In case of the polygon-dependent foliations  $\mathcal{F}_1, \mathcal{F}_2$ , Thurston [57] obtains

$$\Gamma_{\mathcal{F}_a} = \text{vol}(\pi^{-1}(Q)) = 4\pi \cdot \text{Area}(P_a)$$

and thus

- $\mathcal{F}_1$  is cobordant to  $\mathcal{F}_2 \implies \text{Area}(P_1) = \text{Area}(P_2)$
- $\mathcal{F}_1$  and  $\mathcal{F}_2$  are non-cobordant  $\iff \text{Area}(P_1) \neq \text{Area}(P_2)$

We note that  $\text{Area}(P) = (k-2)\pi - \sum_k \alpha_k$ . The Godbillon-Vey class is an element of the deRham cohomology  $H^3(S^3, \mathbb{R})$  which will be used later to construct a relation to gerbes. Furthermore we remark that the classification is not complete. Thurston constructed only a surjective homomorphism from the group of cobordism classes of foliation of  $S^3$  into the real numbers  $\mathbb{R}$ . We remark the close connection between the Godbillon-Vey class (1) and the Chern-Simons form if  $\theta$  can be interpreted as connection of a suitable line bundle.

### 2.3 Codimension-one foliations on 3-manifolds

Now we will discuss the general case of a compact 3-manifold. Later on we will need the codimension-1 foliations of a homology 3-sphere  $\Sigma$ . Because of the diffeomorphism  $\Sigma \# S^3 = \Sigma$ , we can relate a foliation on  $\Sigma$  to a foliation on  $S^3$ . By using the surgery along a knot or link, we are able to construct the codimension-one foliation for every compact 3-manifold.

**Theorem 1** *Given a compact 3-manifold  $\Sigma$  without boundary. Every codimension-one foliation  $\mathcal{F}$  of the 3-sphere  $S^3$  (constructed above) induces a codimension-one foliation  $\mathcal{F}_\Sigma$  on  $\Sigma$ . For every cobordism class  $[\mathcal{F}]$  as element of the deRham cohomology  $H^3(S^3, \mathbb{R})$ , there exists an element of  $H^3(\Sigma, \mathbb{R})$  with a cobordism class  $[\mathcal{F}_\Sigma]$ .*

*Proof* The proof can be found in [9].  $\square$

## 2.4 The smooth holonomy groupoid and its $C^*$ algebra

Let  $(M, F)$  be a foliated manifold. Now we shall construct a von Neumann algebra  $W(M, F)$  canonically associated to  $(M, F)$  and depending only on the Lebesgue measure class on the space  $X = M/F$  of leaves of the foliation. *In the following we will identify the leaf space with this von Neumann algebra.* The classical point of view,  $L^\infty(X)$ , will only give the center  $Z(W)$  of  $W$ . According to Connes [25], we assign to each leaf  $\ell \in X$  the canonical Hilbert space of square-integrable half-densities  $L^2(\ell)$ . This assignment, i.e. a measurable map, is called a random operator forming a von Neumann  $W(M, F)$ . The explicit construction of this algebra can be found in [23]. Here we remark that  $W(M, F)$  is also a noncommutative Banach algebra which is used above. Alternatively we can construct  $W(M, F)$  as the compact endomorphisms of modules over the  $C^*$  algebra  $C^*(M, F)$  of the foliation  $(M, F)$  also known as holonomy algebra. From the point of view of K theory, both algebras  $W(M, F)$  and  $C^*(M, F)$  are Morita-equivalent to each other leading to the same  $K$  groups. In the following we will construct the algebra  $C^*(M, F)$  by using the holonomy groupoid of the foliation.

Given a leaf  $\ell$  of  $(M, F)$  and two points  $x, y \in \ell$  of this leaf, any simple path  $\gamma$  from  $x$  to  $y$  on the leaf  $\ell$  uniquely determines a germ  $h(\gamma)$  of a diffeomorphism from a transverse neighborhood of  $x$  to a transverse neighborhood of  $y$ . The germ of diffeomorphism  $h(\gamma)$  thus obtained only depends upon the homotopy class of  $\gamma$  in the fundamental groupoid of the leaf  $\ell$ , and is called the holonomy of the path  $\gamma$ . The holonomy groupoid of a leaf  $\ell$  is the quotient of its fundamental groupoid by the equivalence relation which identifies two paths  $\gamma$  and  $\gamma'$  from  $x$  to  $y$  (both in  $\ell$ ) iff  $h(\gamma) = h(\gamma')$ . The holonomy covering  $\tilde{\ell}$  of a leaf is the covering of  $\ell$  associated to the normal subgroup of its fundamental group  $\pi_1(\ell)$  given by paths with trivial holonomy. The holonomy groupoid of the foliation is the union  $G$  of the holonomy groupoids of its leaves.

Recall a groupoid  $\mathbf{G}$  is a category where every morphism is invertible. Let  $G_0$  be a set of objects and  $G_1$  the set of morphisms of  $\mathbf{G}$ , then the structure maps of  $\mathbf{G}$  reads as:

$$G_1 \underset{t}{\times_s} G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \xrightarrow{s} G_0 \xrightarrow{e} G_1 \quad (2)$$

where  $m$  is the composition of the composable two morphisms (target of the first is the source of the second),  $i$  is the inversion of an arrow,  $s, t$  the source and target maps respectively,  $e$  assigns the identity to every object. We assume that  $G_{0,1}$  are smooth manifolds and all structure maps are smooth too. We require that the  $s, t$  maps are submersions, thus  $G_1 \underset{t}{\times_s} G_1$  is a manifold as well. These groupoids are called *smooth* groupoids.

Given an element  $\gamma$  of  $G$ , we denote by  $x = s(\gamma)$  the origin of the path  $\gamma$  and its endpoint  $y = t(\gamma)$  with the range and source maps  $t, s$ . An element  $\gamma$  of  $G$  is thus given by two points  $x = s(\gamma)$  and  $y = t(\gamma)$  of  $M$  together with an equivalence

class of smooth paths: the  $\gamma(t)$ ,  $t \in [0, 1]$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ , tangent to the bundle  $F$  (i.e. with  $\frac{d}{dt}\gamma(t) \in F_{\gamma(t)}$ ,  $\forall t \in \mathbb{R}$ ) identifying  $\gamma_1$  and  $\gamma_2$  as equivalent iff the holonomy of the path  $\gamma_2 \circ \gamma_1^{-1}$  at the point  $x$  is the identity. The graph  $G$  has an obvious composition law. For  $\gamma, \gamma' \in G$ , the composition  $\gamma \circ \gamma'$  makes sense if  $s(\gamma) = t(\gamma')$ . The groupoid  $G$  is by construction a (not necessarily Hausdorff) manifold of dimension  $\dim G = \dim V + \dim F$ . We state that  $G$  is a smooth groupoid, the *smooth holonomy groupoid*.

Then the  $C^*$ -algebra  $C_r^*(M, F)$  of the foliation  $(M, F)$  is the  $C^*$ -algebra  $C_r^*(G)$  of the smooth holonomy groupoid  $G$ . For completeness we will present the explicit construction (see [25] sec. II.8). The basic elements of  $C_r^*(M, F)$  are smooth half-densities with compact supports on  $G$ ,  $f \in C_c^\infty(G, \Omega^{1/2})$ , where  $\Omega^{1/2}$  for  $\gamma \in G$  is the one-dimensional complex vector space  $\Omega_x^{1/2} \otimes \Omega_y^{1/2}$ , where  $s(\gamma) = x, t(\gamma) = y$ , and  $\Omega_x^{1/2}$  is the one-dimensional complex vector space of maps from the exterior power  $\Lambda^k F_x$ ,  $k = \dim F$ , to  $\mathbb{C}$  such that

$$\rho(\lambda\nu) = |\lambda|^{1/2} \rho(\nu) \quad \forall \nu \in \Lambda^k F_x, \lambda \in \mathbb{R}.$$

For  $f, g \in C_c^\infty(G, \Omega^{1/2})$ , the convolution product  $f * g$  is given by the equality

$$(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2)$$

Then we define via  $f^*(\gamma) = \overline{f(\gamma^{-1})}$  a  $*$ -operation making  $C_c^\infty(G, \Omega^{1/2})$  into a  $*$ -algebra. For each leaf  $L$  of  $(M, F)$  one has a natural representation of  $C_c^\infty(G, \Omega^{1/2})$  on the  $L^2$  space of the holonomy covering  $\tilde{L}$  of  $L$ . Fixing a base point  $x \in L$ , one identifies  $\tilde{L}$  with  $G_x = \{\gamma \in G, s(\gamma) = x\}$  and defines the representation

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) \xi(\gamma_2) \quad \forall \xi \in L^2(G_x).$$

The completion of  $C_c^\infty(G, \Omega^{1/2})$  with respect to the norm

$$\|f\| = \sup_{x \in M} \|\pi_x(f)\|$$

makes it into a  $C^*$ -algebra  $C_r^*(M, F)$ . Among all elements of the  $C^*$ -algebra, there are distinguished elements, idempotent operators or projectors having a geometric interpretation in the foliation. For later use, we will construct them explicitly (we follow [25] sec. II.8.β closely). Let  $N \subset M$  be a compact submanifold which is everywhere transverse to the foliation (thus  $\dim(N) = \text{codim}(F)$ ). A small tubular neighborhood  $N'$  of  $N$  in  $M$  defines an induced foliation  $F'$  of  $N'$  over  $N$  with fibers  $\mathbb{R}^k$ ,  $k = \dim F$ . The corresponding  $C^*$ -algebra  $C_r^*(N', F')$  is isomorphic to  $C(N) \otimes \mathcal{K}$  with  $\mathcal{K}$  the  $C^*$ -algebra of compact operators. In particular it contains an idempotent  $e = e^2 = e^*$ ,  $e = 1_N \otimes f \in C(N) \otimes \mathcal{K}$ , where  $f$  is a minimal projection in  $\mathcal{K}$ . The inclusion  $C_r^*(N', F') \subset C_r^*(M, F)$  induces an idempotent in  $C_r^*(M, F)$ . Now we consider the range map  $t$  of the smooth holonomy groupoid  $G$  defining via  $t^{-1}(N) \subset G$  a submanifold. Let  $\xi \in C_c^\infty(t^{-1}(N), s^*(\Omega^{1/2}))$  be a section (with compact support) of the bundle of half-density  $s^*(\Omega^{1/2})$  over  $t^{-1}(N)$  so that the support of  $\xi$  is in the diagonal in  $G$  and

$$\int_{t(\gamma)=y} |\xi(\gamma)|^2 = 1 \quad \forall y \in N.$$

Then the equality

$$e(\gamma) = \sum_{\substack{s(\gamma)=s(\gamma') \\ t(\gamma') \in N}} \bar{\xi}(\gamma' \circ \gamma^{-1}) \xi(\gamma')$$

defines an idempotent  $e \in C_c^\infty(G, \Omega^{1/2}) \subset C_r^*(M, F)$ . Thus, such an idempotent is given by a closed curve in  $M$  transversal to the foliation.

## 2.5 Some information about the factor *III* case

In our case of codimension-1 foliations of the 3-sphere with nontrivial Godbillon-Vey invariant we have the result of Hurder and Katok [39]. Then the corresponding von Neumann algebra  $W(S^3, F)$  contains a factor *III* algebra. At first we will give an overview about the factor *III*.

Remember a von Neumann algebra is an involutive subalgebra  $M$  of the algebra of operators on a Hilbert space  $H$  that has the property of being the commutant of its commutant:  $(M')' = M$ . This property is equivalent to saying that  $M$  is an involutive algebra of operators that is closed under weak limits. A von Neumann algebra  $M$  is said to be hyperfinite if it is generated by an increasing sequence of finite-dimensional subalgebras. Furthermore we call  $M$  a factor if its center is equal to  $\mathbb{C}$ . It is a deep result of Murray and von Neumann that every factor  $M$  can be decomposed into 3 types of factors  $M = M_I \oplus M_{II} \oplus M_{III}$ . The factor *I* case divides into the two classes  $I_n$  and  $I_\infty$  with the hyperfinite factors  $I_n = M_n(\mathbb{C})$  the complex square matrices and  $I_\infty = \mathcal{L}(H)$  the algebra of all operators on an infinite-dimensional Hilbert space  $H$ . The hyperfinite *II* factors are given by  $II_1 = \text{Cliff}_{\mathbb{C}}(E)$ , the Clifford algebra of an infinite-dimensional Euclidean space  $E$ , and  $II_\infty = II_1 \otimes I_\infty$ . The case *III* remained mysterious for a long time. Now we know that there are three cases parametrized by a real number  $\lambda \in [0, 1]$ :  $III_0 = R_W$  the Krieger factor induced by an ergodic flow  $W$ ,  $III_\lambda = R_\lambda$  the Powers factor for  $\lambda \in (0, 1)$  and  $III_1 = R_\infty = R_{\lambda_1} \otimes R_{\lambda_2}$  the Araki-Woods factor for all  $\lambda_1, \lambda_2$  with  $\lambda_1/\lambda_2 \notin \mathbb{Q}$ . We remark that all factor *III* cases are induced by infinite tensor products of the other factors. One example of such an infinite tensor space is the Fock space in quantum field theory.

But now we are interested in an explicit construction of a factor *III* von Neumann algebra of a foliation. The interesting example of this situation is given by the Anosov foliation  $F$  of the unit sphere bundle  $V = T_1 S$  of a compact Riemann surface  $S$  of genus  $g > 1$  endowed with its Riemannian metric of constant curvature  $-1$ . In general the manifold  $V$  is the quotient  $V = G/T$  of the semi-simple Lie group  $G = PSL(2, \mathbb{R})$ , the isometry group of the hyperbolic plane  $\mathbb{H}^2$ , by the discrete cocompact subgroup  $T = \pi_1(S)$ , and the foliation  $F$  of  $V$  is given by the orbits of the action by left multiplication on  $V = G/T$  of the subgroup of upper triangular matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad t \in \mathbb{R}$$

The von Neumann algebra  $M = W(V, F)$  of this foliation is the (unique) hyperfinite factor of type  $III_1 = R_\infty$ . In the subsection 2.2 we describe the construction of the codimension-1 foliation on the 3-sphere  $S^3$ . The main ingredient of this construction is the convex polygon  $P$  in the hyperbolic plane  $\mathbb{H}^2$  having curvature  $-1$ . The Reeb components of this foliation of  $S^3$  are represented by a factor  $I_\infty$  algebra and thus do not contribute to the Godbillon-Vey class. Putting all things together we will get



**Theorem 2** *The codimension-1 foliation of the 3-sphere  $S^3$  with non-trivial Godbillon-Vey invariant is also associated to a von Neumann algebra  $W(S^3, F)$  induced by the foliation which contains a factor  $III$  algebra, the hyperfinite  $III_1$  factor  $R_\infty$ .*

*Proof* This theorem follows mostly from the work Hurder and Katok [39]. The codimension-1 foliation of the 3-sphere was constructed in subsection 2.2. It admits a non-trivial Godbillon-Vey invariant related to the volume of the polygon  $P$  in  $\mathbb{H}^2$ . The whole construction do not depend on the number of vertices of  $P$  but on the volume  $vol(P)$  only. Thus without loss of generality, we can choose the even number  $4g$  for  $g \in \mathbb{N}$  of vertices for  $P$ . As model of the hyperbolic plane we choose the usual upper half-plane model where the group  $SL(2, \mathbb{R})$  (the real Möbius transformations) and the hyperbolic group  $PSL(2, \mathbb{R})$  (the group of all orientation-preserving isometries of  $\mathbb{H}^2$ ) act via fractional linear transformations. Then the polygon  $P$  is a fundamental polygon representing a Riemann surface  $S$  of genus  $g$ . Via the procedure above, we can construct a foliation on  $T_1S = PSL(2, \mathbb{R})/T$  with  $T = \pi_1(S)$ . This foliation is also induced from the foliation of  $T_1\mathbb{H}$  (as well as the foliation of the  $S^3$ ) via the left action above. The difference between the foliation on  $T_1S$  and on  $S^3$  is given by the different usage of the polygon  $P$ . Thus the von Neumann algebra  $W(S^3, F)$  of the codimension-1 foliation of the 3-sphere contains a factor  $III$  algebra in agreement with the results in [39]. In the notation above we have the unit tangent bundle  $T_1P$  of the polygon  $P$  equipped with an Anosov foliation (see also [52]). The group  $PSL(2, \mathbb{R})$  acts as isometry on  $\mathbb{H}^2$  where the modular group  $PSL(2, \mathbb{Z})$  acts as discrete subgroup leaving the polygon  $P$  (seen as fundamental domain) invariant. The upper triangular matrices above are elements of  $PSL(2, \mathbb{R})$  and act by linear fractional transformation inducing a shift. The orbits of this action have therefore constant velocity (the horocycle flow) and we are done.  $\square$

We showed that this factor  $III$  algebra is the hyperfinite  $III_1$  factor  $R_\infty$ . Now one may ask, what is the physical meaning of the factor  $III$ ? Because of the Tomita-Takesaki-theory, factor  $III$  algebras are deeply connected to the characterization of equilibrium temperature states of quantum states in statistical mechanics and field theory also known as Kubo-Martin-Schwinger (KMS) condition. Furthermore in the quantum field theory with local observables (see Borchers [14] for an overview) one obtains close connections to Tomita-Takesaki-theory. For instance one was able to show that on the vacuum Hilbert space with one vacuum vector the algebra of local observables is a factor  $III_1$  algebra. As shown by Thiemann et. al. [43] on a class of diffeomorphism invariant theories there exists an unique vacuum vector. Thus the observables algebra must be of this type.

### 3 Exotic $\mathbb{R}^4$ and codimension-one foliations

Einstein's insight that gravity is the manifestation of geometry leads to a new view on the structure of spacetime. From the mathematical point of view, spacetime is a smooth 4-manifold endowed with a (smooth) metric as basic variable for general relativity. Later on, the existence question for Lorentz structure and causality problems (see Hawking and Ellis [38]) gave further restrictions on the 4-manifold: causality implies non-compactness, Lorentz structure needs a codimension-1 foliation. Usually, one starts with a globally foliated, non-compact 4-manifold  $\Sigma \times \mathbb{R}$  fulfilling all restrictions where  $\Sigma$  is a smooth 3-manifold representing the spatial part. But other non-compact 4-manifolds are also possible, i.e. it is enough to assume a non-compact, smooth 4-

manifold endowed with a codimension-1 foliation. All these restrictions on the representation of spacetime by the manifold concept are clearly motivated by physical questions. Among the properties there is one distinguished element: the smoothness. Usually one assumes a smooth, unique atlas of charts (i.e. a smooth or differential structure) covering the manifold where the smoothness is induced by the unique smooth structure on  $\mathbb{R}$ . But that is not the full story. Even in dimension 4, there are an infinity of possible other smoothness structures (i.e. a smooth atlas) non-diffeomorphic to each other. For a deeper insight we refer to the book [8].

### 3.1 Smoothness on manifolds

If two manifolds are homeomorphic but non-diffeomorphic, they are **exotic** to each other. The smoothness structure is called an **exotic smoothness structure**.

The implications for physics are tremendous because we rely on the smooth calculus to formulate field theories. Thus different smoothness structures have to represent different physical situations leading to different measurable results. But it should be stressed that *exotic smoothness is not exotic physics*. Exotic smoothness is a mathematical structure which should be further explored to understand its physical relevance.

Usually one starts with a topological manifold  $M$  and introduces structures on them. Then one has the following ladder of possible structures:

$$\begin{aligned} \text{Topology} &\rightarrow \text{piecewise-linear(PL)} \rightarrow \text{Smoothness} \rightarrow \\ &\rightarrow \text{bundles, Lorentz, Spin etc.} \rightarrow \text{metric, geometry, ...} \end{aligned}$$

We do not want to discuss the first transition, i.e. the existence of a triangulation on a topological manifold. But we remark that the existence of a PL structure implies uniquely a smoothness structure in all dimensions smaller than 7 [41]. The following basic facts should the reader keep in mind for any  $n$ -dimensional manifold  $M^n$ :

1. The maximal differentiable atlas  $\mathcal{A}$  of  $M^n$  is the smoothness structure.
2. Every manifold  $M^n$  can be embedded in  $\mathbb{R}^N$  with  $N > 2n$ . A smooth embedding  $M^n \hookrightarrow \mathbb{R}^N$  induces the *standard smooth structure* on  $M$ . All other possible smoothness structures are called exotic smoothness structures.
3. The existence of a smoothness structure is *necessary* to introduce Riemannian or Lorentz structures on  $M$ , but the smoothness structure don't further restrict the Lorentz structure.

### 3.2 Small exotic $\mathbb{R}^4$ 's and Akbulut corks

Now we consider two homeomorphic, smooth, but non-diffeomorphic 4-manifolds  $M_0$  and  $M$ . As expressed above, a comparison of both smoothness structures is given by a h-cobordism  $W$  between  $M_0$  and  $M$  ( $M, M_0$  are homeomorphic). Let the 4-manifolds additionally be compact, closed and simple-connected, then we have the structure theorem<sup>2</sup> of h-cobordisms [26]:

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<sup>2</sup> A diffeomorphism will be described by the symbol  $=$  in the following.

**Theorem 3** *Let  $W$  be a h-cobordism between  $M_0$  and  $M$ , then there are contractible submanifolds  $A_0 \subset M_0$ ,  $A \subset M$  and a h-subcobordism  $X \subset W$  with  $\partial X = A_0 \sqcup A$ , so that the remaining h-cobordism  $W \setminus X$  trivializes  $W \setminus X = (M_0 \setminus A_0) \times [0, 1]$  inducing a diffeomorphism between  $M_0 \setminus A_0$  and  $M \setminus A$ .*

In short it means that the smoothness structure of  $M$  is determined by the contractible manifold  $A$  – its Akbulut cork – and by the embedding of  $A$  into  $M$ . As shown by Freedman[29], the Akbulut cork has a homology 3-sphere<sup>3</sup> as boundary. The embedding of the cork can be derived now from the structure of the h-subcobordism  $X$  between  $A_0$  and  $A$ . For that purpose we cut  $A_0$  out from  $M_0$  and  $A$  out from  $M$ . Then we glue in both submanifolds  $A_0, A$  via the maps  $\tau_0 : \partial A_0 \rightarrow \partial(M_0 \setminus A_0) = \partial A_0$  and  $\tau : \partial A \rightarrow \partial(M \setminus A) = \partial A$ . Both maps  $\tau_0, \tau$  are involutions, i.e.  $\tau \circ \tau = id$ . One of these maps (say  $\tau_0$ ) can be chosen to be trivial (say  $\tau_0 = id$ ). Thus the involution  $\tau$  determines the smoothness structure. Especially the topology of the Akbulut cork  $A$  and its boundary  $\partial A$  is given by the topology of  $M$ . For instance, the Akbulut cork of the blow-uped 4-dimensional  $K3$  surface  $K3 \# \overline{\mathbb{C}P}^2$  is the so-called *Mazur manifold* [2, 1] with the *Brieskorn-Sphere*  $B(2, 5, 7)$  as boundary. Akbulut and its coworkers [3, 4] discuss many examples of Akbulut corks and the dependence of the smoothness structure on the cork.

For the following we need a short account of the proof of the h-cobordism structure theorem. The interior of every h-cobordism can be divided into pieces, called handle [44]. A  $k$ -handle is the manifold  $D^k \times D^{5-k}$  which will be glued along the boundary  $S^k \times D^{5-k}$ . The pairs of  $0-1$ - and  $4-5$ -handles in a h-cobordism between the two homeomorphic 4-manifolds  $M_0$  and  $M$  can be killed by a general procedure ([44], §8). Thus only the pairs of  $2-3$ -handles are left. Exactly these pairs are the difference between the smooth h-cobordism and the topological h-cobordism. To eliminate the  $2-3$ -handles one has to embed a disk without self-intersections into  $M$  (Whitney trick). But that is mostly impossible in 4-dimensional manifolds. Therefore Casson [21] constructed by an infinite, recursive process a special handle – the Casson-handle  $CH$  – containing the required disk without self-intersections. Freedman was able to show topologically the existence of this disk and he constructs a homeomorphism between every Casson handle  $CH$  and the open 2-handle  $D^2 \times \mathbb{R}^2$  [29]. But  $CH$  is in general non-diffeomorphic to  $D^2 \times \mathbb{R}^2$  as shown later by Gompf [34, 35].

Now we consider the smooth h-cobordism  $W$  together with a neighborhood  $N$  of  $2-3$ -handles. It is enough to assume a pair of handles with two self-intersections (of opposite orientation) between the 2- and 3-Spheres at the boundary of the handle. Thus one can construct an Akbulut cork  $A$  in  $M$  out of this data [26]. The pair of  $2-3$ -handles can be eliminated topologically by the embedding of a Casson handle. Then as shown by Bizaca and Gompf [13] the neighborhood  $N$  of the handle pair as well the neighborhood  $N(A)$  of the embedded Akbulut cork consists of the cork  $A$  and the Casson handle  $CH$ . Especially the open neighborhood  $N(A)$  of the Akbulut cork is an exotic  $\mathbb{R}^4$ . The situation was analyzed in [36]:

**Theorem 4** *Let  $W^5$  be a non-trivial (smooth) h-cobordism between  $M_0^4$  and  $M^4$  (i.e.  $W$  is not diffeomorphic to  $M \times [0, 1]$ ). Then there is an open sub-h-cobordism  $U^5$  that is homeomorphic to  $\mathbb{R}^4 \times [0, 1]$  and contains a compact contractible sub-h-cobordism  $X$  (the cobordism between the Akbulut corks, see above), such that both  $W$  and  $U$  are*

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<sup>3</sup> A homology 3-sphere is a 3-manifold with the same homology as the 3-sphere  $S^3$ .

trivial cobordisms outside of  $X$ , i.e. one has the diffeomorphisms

$$W \setminus X = ((W \cap M) \setminus X) \times [0, 1] \quad \text{and} \quad U \setminus X = ((U \cap M) \setminus X) \times [0, 1]$$

(the latter can be chosen to be the restriction of the former). Furthermore the open sets  $U \cap M$  and  $U \cap M_0$  are homeomorphic to  $\mathbb{R}^4$  which are exotic  $\mathbb{R}^4$  if  $W$  is non-trivial.

Then one gets an exotic  $\mathbb{R}^4$  which smoothly embeds automatically in the 4-sphere, called a small exotic  $\mathbb{R}^4$ . Furthermore we remark that the exoticness of the  $\mathbb{R}^4$  is connected with the non-trivial smooth h-cobordism  $W^5$ , i.e. the failure of the smooth h-cobordism theorem implies the existence of small exotic  $\mathbb{R}^4$ s.

### 3.3 Exotic $\mathbb{R}^4$ and Casson handles

The theorem 4 relates a non-trivial h-cobordism between two compact, simple-connected, smooth 4-manifolds to a small exotic  $\mathbb{R}^4$ . Using theorem 3, we can understand where the non-triviality of the h-cobordism comes from: one of the Akbulut corks, say  $A$ , must be glued in by using a non-trivial involution of the boundary  $\partial A$ . In the notation above, there is a non-product h-cobordism  $W$  between  $M^4$  and  $M_0^4$  with a h-subcobordism  $X$  between  $A_0 \subset M_0$  and  $A \subset M$ . There is an open neighborhood  $U$  of the h-subcobordism  $X$  which is an open h-cobordism  $U$  between the open neighborhoods  $N(A) \subset M$ ,  $N(A_0) \subset M_0$ . Both neighborhoods are homeomorphic to  $\mathbb{R}^4$  but not diffeomorphic to the standard  $\mathbb{R}^4$  (as induced from the non-productness of the h-cobordism  $W$ ). This exotic  $\mathbb{R}^4$  is the interior of the attachment of a Casson handle  $CH$  to the boundary  $\partial A$  of the cork  $A$ .

Now let us consider the basic construction of the Casson handle  $CH$ . Let  $M$  be a smooth, compact, simple-connected 4-manifold and  $f : D^2 \rightarrow M$  a (codimension-2) mapping. By using diffeomorphisms of  $D^2$  and  $M$ , one can deform the mapping  $f$  to get an immersion (i.e. injective differential) generically with only double points (i.e.  $\#|f^{-1}(f(x))| = 2$ ) as singularities [33]. But to incorporate the generic location of the disk, one is rather interesting in the mapping of a 2-handle  $D^2 \times D^2$  induced by  $f \times id : D^2 \times D^2 \rightarrow M$  from  $f$ . Then every double point (or self-intersection) of  $f(D^2)$  leads to self-plumbings of the 2-handle  $D^2 \times D^2$ . A self-plumbing is an identification of  $D_0^2 \times D^2$  with  $D_1^2 \times D^2$  where  $D_0^2, D_1^2 \subset D^2$  are disjoint sub-disks of the first factor disk<sup>4</sup>. Consider the pair  $(D^2 \times D^2, \partial D^2 \times D^2)$  and produce finitely many self-plumbings away from the attaching region  $\partial D^2 \times D^2$  to get a kinky handle  $(k, \partial^- k)$  where  $\partial^- k$  denotes the attaching region of the kinky handle. A kinky handle  $(k, \partial^- k)$  is a one-stage tower  $(T_1, \partial^- T_1)$  and an  $(n+1)$ -stage tower  $(T_{n+1}, \partial^- T_{n+1})$  is an  $n$ -stage tower union kinky handles  $\bigcup_{\ell=1}^n (T_\ell, \partial^- T_\ell)$  where two towers are attached along  $\partial^- T_\ell$ . Let  $T_n^-$  be  $(\text{interior } T_n) \cup \partial^- T_n$  and the Casson handle

$$CH = \bigcup_{\ell=0} T_\ell^-$$

is the union of towers (with direct limit topology induced from the inclusions  $T_n \hookrightarrow T_{n+1}$ ). A Casson handle is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point  $*$ , having all edge paths

<sup>4</sup> In complex coordinates the plumbing may be written as  $(z, w) \mapsto (w, z)$  or  $(z, w) \mapsto (\bar{w}, \bar{z})$  creating either a positive or negative (respectively) double point on the disk  $D^2 \times 0$  (the core).

infinitely extendable away from  $\star$ . Each edge should be given a label  $+$  or  $-$ . Here is the construction:  $\text{tree} \rightarrow CH$ . Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. The simplest non-trivial Casson handle is represented by the tree  $Tree_+$ : each level has one branching point with positive sign  $+$ .

Given a labeled based tree  $Q$ , let us describe a subset  $U_Q$  of  $D^2 \times D^2$ . Now we will construct a  $(U_Q, \partial D^2 \times D^2)$  which is diffeomorphic to the Casson handle associated to  $Q$ . In  $D^2 \times D^2$  embed a ramified Whitehead link with one Whitehead link component for every edge labeled by  $+$  leaving  $\star$  and one mirror image Whitehead link component for every edge labeled by  $-$  (minus) leaving  $\star$ . Corresponding to each first level node of  $Q$  we have already found a (normally framed) solid torus embedded in  $D^2 \times \partial D^2$ . In each of these solid tori embed a ramified Whitehead link, ramified according to the number of  $+$  and  $-$  labeled branches leaving that node. We can do that process for every level of  $Q$ . Let the disjoint union of the (closed) solid tori in the  $n$ th family (one solid torus for each branch at level  $n$  in  $Q$ ) be denoted by  $X_n$ .  $Q$  tells us how to construct an infinite chain of inclusions:

$$\dots \subset X_{n+1} \subset X_n \subset X_{n-1} \subset \dots \subset X_1 \subset D^2 \times \partial D^2$$

and we define the Whitehead decomposition  $Wh_Q = \bigcap_{n=1}^{\infty} X_n$  of  $Q$ .  $Wh_Q$  is the Whitehead continuum [61] for the simplest unbranched tree. We define  $U_Q$  to be

$$U_Q = D^2 \times D^2 \setminus (D^2 \times \partial D^2 \cup \text{closure}(Wh_Q))$$

alternatively one can also write

$$U_Q = D^2 \times D^2 \setminus \text{cone}(Wh_Q) \quad (3)$$

where  $\text{cone}()$  is the cone of a space

$$\text{cone}(A) = A \times [0, 1] / (x, 0) \sim (x', 0) \quad \forall x, x' \in A$$

over the point  $(0, 0) \in D^2 \times D^2$ . As Freedman (see [29] Theorem 2.2) showed  $U_Q$  is diffeomorphic to the Casson handle  $CH_Q$  given by the tree  $Q$ .

### 3.4 The design of a Casson handle and its foliation

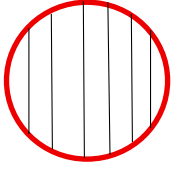
A Casson handle is represented by a labeled finitely-branching tree  $Q$  with base point  $\star$ , having all edge paths infinitely extendable away from  $\star$ . Each edge should be given a label  $+$  or  $-$  and each vertex corresponds to a kinky handle where the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The open handle  $D^2 \times \mathbb{R}^2$  is represented by the  $\star$ , i.e. there are no kinky handles. One of the cornerstones of Freedmans proof of the homeomorphism between a Casson handle  $CH$  and the open 2-handle  $H = D^2 \times \mathbb{R}^2$  are the reembedding theorems. Then one foliates  $CH$  and  $H$  by copies of the frontier  $Fr(CH)$ . The frontier of a set  $K$  is defined by  $Fr(K) = \text{closure}(\text{closure}(K) \setminus K)$ . As example we consider the interior  $\text{int}(D^2)$  of a disk and obtain for the frontier  $Fr(\text{int}(D^2)) = \text{closure}(\text{closure}(\text{int}(D^2)) \setminus \text{int}(D^2)) = \partial D^2$ ,

i.e. the boundary of the disk  $D^2$ . Then Freedman ([29] p.398) constructs another labeled tree  $S(Q)$  from the tree  $Q$ . There is a base point from which a single edge (called “decimal point”) emerges. The tree is binary: one edge enters and two edges leaving a vertex (or every vertex is trivalent). The edges are named by initial segments of infinite base 3-decimals representing numbers in the standard “middle third” Cantor set<sup>5</sup>  $CS \subset [0, 1]$ . Each edge  $e$  of  $S(Q)$  carries a label  $\tau_e$  where  $\tau_e$  is an ordered finite disjoint union of 5-stage towers together with an ordered collection of standard loops generating the fundamental group. There is three constraints on the labels which leads to the correspondence between the  $\pm$  labeled tree  $Q$  and the (associated)  $\tau$ -labeled tree  $S(Q)$ . One calls  $S(Q)$  the design.

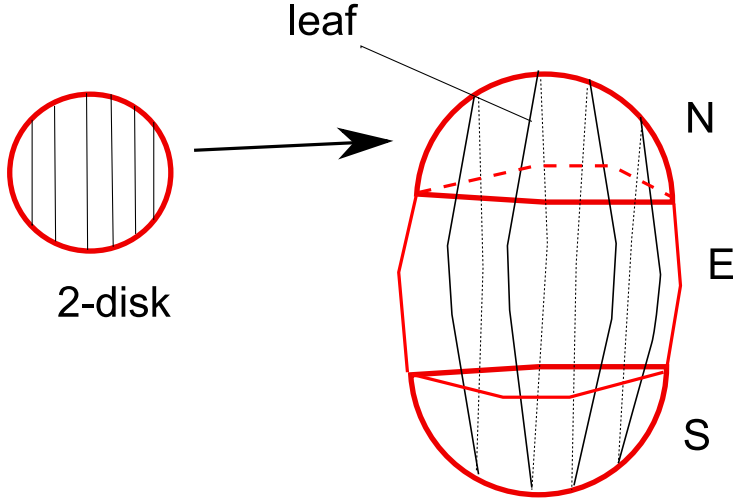
Two words are in order for the design  $S(Q)$ : first, every sequence of 0’s and 2’s is one path in  $S(Q)$  representing one embedded Casson handle  $CH_{Q_1} \subset CH_Q$  where both trees are related like  $Q \subset Q_1$ . For example, the Casson handle corresponding to .020202... is obtained as the union of the 5-stage towers  $T^0 \cup T^{02} \cup T^{020} \cup T^{0202} \cup T^{02020} \cup T^{020202} \cup \dots$ . For later usage we identify the sequence .00000... with the Tree  $Tree_+$ . Secondly, there are gaps, i.e. we have only a Cantor set of Casson handles not a continuum. For instance a gap is lying between the paths .022222... and .20000... In the proof of Freedman, the gaps are shrunk to a point and one gets the desired homeomorphism. Here we will use this structure to produce a foliation of the design. Every path in  $S(Q)$  is represented by one sequence over the alphabet  $\{0, 2\}$ . Every gap is a sequence containing at least one 1 (so for instance .1222... or .012222...). There is now a natural order structure given by the sequence (for instance .022222... < .12222... < .22222...). The leaves are the corresponding gaps or Casson handles (represented by the union 5-stage towers ending with  $T^{02222...}$ ,  $T^{12222...}$  or  $T^{22222...}$ ). The tree structure of the design  $S(Q)$  should be also reflected in the foliation to represent every path in  $S(Q)$  as a union of 5-stage towers. By the reembedding theorems, the 5-stage towers can be embedded into each other. Then we obtain two foliations of the (topological) open 2-handle  $D^2 \times \mathbb{R}^2$ : a codimension-1 foliation along one  $\mathbb{R}$ -axis labeled by the sequences (for instance .022222... < .12222... < .22222...) and a second codimension-1 foliation along the radius of the disk  $D^2$  induced by inclusion of the 5-stage towers (for instance  $T^0 \supset T^{02} \supset T^{020} \supset \dots$ ). Especially the exploration of a Casson handle by using the design is given by its frontier, in this case, minus the attaching region. In case of a usual tower we get the frontier  $S^1 \times D^2/Wh_\gamma$  with  $\gamma \in S(Q)$ . The gaps have a similar structure. Then the foliation of the Casson handle (induced from the design) is given by the leaves  $S^1 \times D^1$  over the disk  $D^2$  in the Casson handle, i.e. the disk  $D^2$  is foliated by parallel lines (see Fig. 1). So, every Casson handle with a given tree  $Q$  has a codimension-one foliation given by its design.

This foliation can be also understood as a foliated cobordism. For that purpose we consider the foliation as part of a foliation of the 2-sphere (see Fig. 2). The 2-sphere is decomposed by  $S^2 = N \cup E \cup S$ , two pole regions  $N, S$  ( $N, S = D^2$ ) and an equator region  $E = S^1 \times D^1$ . The foliation of the disk as in Fig. 1 can be used to foliate  $N$  and  $S$ . Both foliations can be connected by the leaves  $S^1$  which are the longitudes. Then

<sup>5</sup> This kind of Cantor set is given by the following construction: Start with the unit Interval  $S_0 = [0, 1]$  and remove from that set the middle third and set  $S_1 = S_0 \setminus (1/3, 2/3)$ . Continue in this fashion, where  $S_{n+1} = S_n \setminus \{\text{middle thirds of subintervals of } S_n\}$ . Then the Cantor set  $C.s.$  is defined as  $C.s. = \bigcap_n S_n$ . With other words, if we using a ternary system (a number system with base 3), then we can write the Cantor set as  $C.s. = \{x : x = (0.a_1a_2a_3\dots) \text{ where each } a_i = 0 \text{ or } 2\}$ .



**Figure 1** foliation of the disk  $D^2$  in the design  $S(Q)$



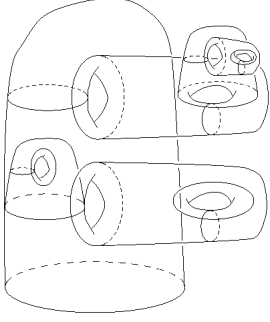
**Figure 2** foliation of the 2-sphere as foliated cobordism  $E$  of the two disks  $N, S$

one obtains a foliated cobordism between  $N$  and  $S$  given by the obvious foliation of the equator region  $E$  (a cylinder).

### 3.5 Capped gropes and its design

In this subsection we discuss a possible generalization of Casson handles. The modern way to the classification of 4-manifolds used “capped gropes”, a mixed variant of Casson handle and grope (chapters 1 to 4 in [28]). We do not want to complicate the situation more than needed. But for later developments we have to discuss some part of the theory but we remark that all results can be easily generalized to capped gropes as well.

A grope is a special pair  $(2\text{-complex}, \text{circle})$ , where the circle is referred to as the boundary of the grope. There is an anomalous case when the depth is 1: the unique grope of depth 1 is the pair  $(\text{circle}, \text{circle})$ . A grope of depth 2 is a punctured surface with the boundary circle specified (see Fig. 3). To form a grope  $G$  of depth  $n$ , take a punctured surface,  $F$ , and prescribe a symplectic basis  $\{\alpha_i, \beta_j\}$ . That is,  $\alpha_i$  and  $\beta_j$  are



**Figure 3** Example of a grope with symplectic basis as curves around the holes

embedded curves in  $F$  which represent a basis of  $H_1(F)$  such that the only intersections among the  $\alpha_i$  and  $\beta_j$  occur when  $\alpha_i$  and  $\beta_j$  meet in a single point  $\alpha_i \cdot \beta_j = 1$ . Now glue gropes of depth  $< n$  along their boundary circles to each  $\alpha_i$  and  $\beta_j$  with at least one such added grope being of depth  $n - 1$ . (Note that we are allowing any added grope to be of depth 1, in which case we are not really adding a grope.) The surface  $F \subset G$  is called the bottom stage of the grope and its boundary is the boundary of the grope. The tips of the grope are those symplectic basis elements of the various punctured surfaces of the grope which do not have gropes of depth  $> 1$  attached to them.

**Definition 1** A capped grope is a grope with disks (the caps) attached to all its tips. The grope without the caps is sometimes called the body of the capped grope.

The capped grope (as cope) was firstly described by Freedman in 1983[30]. The caps are only immersed disks like in case of the Casson handle to make the grope simply-connected. The great advantage is the simpler frontier, instead of  $S^1 \times D^2 / Wh_\gamma$  (see subsection 3.4) one has solid tori  $S^1 \times D^2$  as frontier of the capped grope (as shown in [5]). The corresponding design (and its parametrization) can be described similar to the Casson handle by sequences containing 0 and 2 (see section 4.5 in [28]). There are also gaps (described by a 1 in the sequence) who look like  $S^1 \times D^3$ .

### 3.6 The radial family of uncountable-many small exotic $\mathbb{R}^4$

Given a small exotic  $\mathbb{R}^4$   $R$  induced from the non-product h-cobordism  $W$  between  $M$  and  $M_0$  with Akbulut corks  $A \subset M$  and  $A_0 \subset M_0$ , respectively. Let  $K \subset \mathbb{R}^4$  be a compact subset. Bizaca and Gompf [13] constructed the small exotic  $\mathbb{R}_1^4$  by using the simplest tree  $Tree_+$ . Bizaca [11,12] showed that the Casson handle generated by  $Tree_+$  is an exotic Casson handle. Using Theorem 3.2 of [27], there is a topological radius function  $\rho : \mathbb{R}_1^4 \rightarrow [0, +\infty)$  (polar coordinates) so that  $\mathbb{R}_t^4 = \rho^{-1}([0, r))$  with  $t = 1 - \frac{1}{r}$ . Then  $K \subset \mathbb{R}_0^4$  and  $\mathbb{R}_t^4$  is also a small exotic  $\mathbb{R}^4$  for  $t$  belonging to a Cantor set  $CS \subset [0, 1]$ . Especially two exotic  $\mathbb{R}_s^4$  and  $\mathbb{R}_t^4$  are non-diffeomorphic for  $s < t$  except for countable many pairs. In [27] it was claimed that there is a smoothly embedded homology 4-disk  $A$ . The boundary  $\partial A$  is a homology 3-sphere with a non-trivial representation of its fundamental group into  $SO(3)$  (so  $\partial A$  cannot be diffeomorphic to a



3-sphere). According to Theorem 4 this homology 4-disk must be identified with the Akbulut cork of the non-trivial h-cobordism. The cork  $A$  is contractable and can be (at least) build by one 1-handle and one 2-handle (case of a Mazur manifold). Given a radial family  $\mathbb{R}_t^4$  with radius  $r = \frac{1}{1-t}$  so that  $t = 1 - \frac{1}{r} \in CS \subset [0, 1]$ . Suppose there is a diffeomorphism

$$(d, id_K) : (\mathbb{R}_s^4, K) \rightarrow (\mathbb{R}_t^4, K) \quad s \neq t \in CS$$

fixing the compact subset  $K$ . Then this map  $d$  induces end-periodic manifolds<sup>6</sup>  $M \setminus (\bigcap_{i=0}^{\infty} d^i(\mathbb{R}_s^4))$  and  $M_0 \setminus (\bigcap_{i=0}^{\infty} d^i(\mathbb{R}_s^4))$  which must be smoothable contradicting a theorem of Taubes [53]. Therefore  $\mathbb{R}_s^4$  and  $\mathbb{R}_t^4$  are non-diffeomorphic for  $t \neq s$  (except for countable many possibilities).

### 3.7 Exotic $\mathbb{R}^4$ and codimension-1 foliations

In this subsection we will construct a codimension-one foliation on the boundary  $\partial A$  of the cork with non-trivial Godbillon-Vey invariant. The strategy of the proof goes like this: we use the foliation of the design of the Casson handle (see subsection 3.4) for the radial family  $\mathbb{R}_t^4$  to induce a foliated cobordism  $\partial A \times [0, 1]$ . The restriction to its boundary gives cobordant codimension-1 foliations of  $\partial A$  with non-trivial Godbillon-Vey invariant  $r^2 = \frac{1}{(1-t)^2}$ . In the subsection 2.2 we described a foliation of the 3-sphere unique up to foliated cobordism for every given value of the Godbillon-Vey invariant. By theorem 1, we get a corresponding foliation on  $\partial A$  (with the same Godbillon-Vey invariant). Finally we obtain:

**Theorem 5** *Given a radial family  $R_t$  of small exotic  $\mathbb{R}_t^4$  with radius  $r$  and  $t = 1 - \frac{1}{r} \in CS \subset [0, 1]$  induced from the non-product h-cobordism  $W$  between  $M$  and  $M_0$  with Akbulut cork  $A \subset M$  and  $A \subset M_0$ , respectively. The radial family  $R_t$  determines a family of codimension-one foliations of  $\partial A$  with Godbillon-Vey invariant  $r^2$ . Furthermore given two exotic spaces  $R_t$  and  $R_s$ , homeomorphic but non-diffeomorphic to each other (and so  $t \neq s$ ) then the two corresponding codimension-one foliation of  $\partial A$  are non-cobordant to each other.*

*Proof* The proof can be found in [9] and in the appendix A.  $\square$

In the theorem 5 above we constructed a relation between codimension-1 foliations on  $\Sigma = \partial A$  and the radius for the radial family of small exotic  $\mathbb{R}^4$ . By using theorem 1 we can trace back the foliation on  $\Sigma$  by a foliation on the 3-sphere  $S^3$ . This situation can be seen differently by using the diffeomorphism  $\Sigma = \Sigma \# S^3$ . Then, the foliation on  $S^3$  induces a foliation on  $\Sigma$  at least partially. Thus, we have a 3-sphere  $S^3$  lying at the boundary  $\partial A = \Sigma$  of the Akbulut cork  $A$  inducing a codimension-1 foliation on  $\Sigma$ . Then by theorem 5:

**Corollary 1** *Any class in  $H^3(S^3, \mathbb{R})$  induces a small exotic  $\mathbb{R}^4$  where  $S^3$  lies at the boundary  $\Sigma = \partial A$  of the cork  $A$ .*

---

<sup>6</sup> We ignore the inclusion for simplicity.

### 3.8 Integer Godbillon-Vey invariants and flat bundles

Clearly the integer classes  $H^3(S^3, \mathbb{Z}) \subset H^3(S^3, \mathbb{R})$  are a subset of the full set and one can use the construction above to get the foliation. Especially the polygon  $P$  must be formed by segments with angles  $\alpha_k$  of integer value with respect to  $\pi$  to get an integer value for the volume  $\text{Area}(P) = (k-2)\pi - \sum_k \alpha_k$  up to a  $\pi$ -factor. Using the work of Goldman and Brooks [18], one can construct a foliation admitting an integer Godbillon-Vey invariant. The corresponding foliation is induced by the unit tangent  $T_1\mathbb{H}^2$  or by the action of the Möbius group  $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2$  (Remark:  $PSL(2, \mathbb{R})$  acts transitively on  $T_1\mathbb{H}^2$  and so we can identify both spaces). The unit tangent bundle  $T_1\mathbb{H}^2 = PSL(2, \mathbb{R})$  is a circle bundle over  $\mathbb{H}^2$  and we can construct the universal cover, a real line bundle over  $\mathbb{H}^2$ , denoted by  $\widetilde{SL}(2, \mathbb{R})$ . In subsection 2.2 we described Thurston's construction of a codimension-1 foliation  $\mathcal{F}$ . In an intermediate step one has the manifold  $M = (S^2 \setminus \{k \text{ punctures}\}) \times S^1$  (with a foliation  $\mathcal{F}$ ). This foliation  $\mathcal{F}$  is defined by a one-form  $\omega$  together with two other 1-forms  $\theta, \eta$  with

$$d\omega = \theta \wedge \omega, \quad d\theta = \omega \wedge \eta, \quad d\eta = \eta \wedge \theta \quad (4)$$

and Godbillon-Vey invariant  $GV(\mathcal{F}) = \theta \wedge d\theta = \omega \wedge \eta \wedge \theta$ . Now we show that the Godbillon-Vey invariant of this foliation  $\mathcal{F}$  is an integer 3-form:

**Lemma 1** *Given a manifold  $M$  with non-trivial fundamental group  $\pi_1(M)$  with foliation  $\mathcal{F}$  defined by the 1-form  $\omega$  together with two 1-forms  $\theta, \eta$  fulfilling the relations (4). If  $M$  can be written as a flat  $PSL(2, \mathbb{R})$ -bundle over a manifold  $N$  with fiber  $S^1$  and  $\pi_1(N) \neq 0$ . Then the pairing of the Godbillon-Vey invariant with the fundamental class  $[M] \in H_3(M)$  is given by*

$$\langle GV(\mathcal{F}), [M] \rangle = \int_M GV(\mathcal{F}) = (4\pi)^2 \cdot \chi(N) \quad (5)$$

with the Euler characteristics  $\chi(M)$  of  $N$ . Up to a normalization constant one obtains an integer value.

*Proof* The proof can be found in [9].  $\square$

Using this lemma we are able to obtain the special foliation (a la Thurston) of the  $S^3$  with integer Godbillon-Vey invariant.

**Theorem 6** *Every  $PSL(2, \mathbb{R})$  flat bundle over  $M = (S^2 \setminus \{k \text{ punctures}\}) \times S^1$  defines a codimension-1 foliation of  $M$  by the horizontal distribution of the flat connection so that its (normalized) Godbillon-Vey invariant is an integer given by*

$$\frac{1}{(4\pi)^2} \langle GV(\mathcal{F}), [M] \rangle = \pm \chi(N) = \pm(2-k) \quad . \quad (6)$$

*This foliation can be extended to the whole 3-sphere  $S^3$  defining an integer class in  $H^3(S^3, \mathbb{Z})$ .*

*Proof* The proof can be found in [9]. The sign of the integral depends on the orientation of the manifold  $M$ .  $\square$

It is an important consequence of the work [18] that the foliation  $\mathcal{F}$  (and its induced counterpart for the 3-sphere  $S^3$ ) is rigid, i.e. a disturbance (or continuous variation) does not change the Godbillon-Vey invariant.

Foliation	Operator algebra
leaf	operator
closed curve transversal to foliation	projector (idempotent operator)
holonomy	linear functional (state)
local chart	center of algebra

**Table 1** relation between foliation and operator algebra

### 3.9 From exotic smoothness to operator algebras

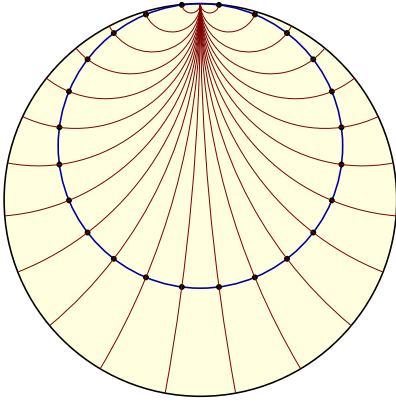
In subsection 2.4 we constructed (following Connes [25]) the smooth holonomy groupoid of a foliation  $F$  and its operator algebra  $C_r^*(M, F)$ . The correspondence between a foliation and the operator algebra (as well as the von Neumann algebra) is visualized by table 1. As extract of our previous paper [9], we obtained a relation between exotic  $\mathbb{R}^4$ 's and codimension-1 foliations of the 3-sphere  $S^3$ . For a codimension-1 foliation there is the Godbillon-Vey invariant [31] as element of  $H^3(M, \mathbb{R})$ . Hurder and Katok [39] showed that the  $C^*$ -algebra of a foliation with non-trivial Godbillon-Vey invariant contains a factor  $III$  subalgebra (by the Anosov-like foliation). Using Tomita-Takesaki-theory, one has a continuous decomposition (as crossed product) of any factor  $III$  algebra  $M$  into a factor  $II_\infty$  algebra  $N$  together with a one-parameter group<sup>7</sup>  $(\theta_\lambda)_{\lambda \in \mathbb{R}_+^*}$  of automorphisms  $\theta_\lambda \in \text{Aut}(N)$  of  $N$ , i.e. one obtains

$$M = N \rtimes_{\theta} \mathbb{R}_+^* .$$

But that means, there is a foliation induced from the foliation of the  $S^3$  producing this  $II_\infty$  factor. As we saw in subsection 2.5 one has a codimension-1 foliation  $F$  as part of the foliation of the  $S^3$  whose von Neumann algebra is the hyperfinite factor  $III_1$ . Connes [25] (in section I.4 page 57ff) constructed the foliation  $F'$  canonically associated to  $F$  having the factor  $II_\infty$  as von Neumann algebra. In our case it is the horocycle flow: Let  $P$  the polygon on the hyperbolic space  $\mathbb{H}^2$  determining the foliation of the  $S^3$  (see subsection ).  $P$  is equipped with the hyperbolic metric  $2|dz|/(1 - |z|^2)$  together with the collection  $T_1P$  of unit tangent vectors to  $P$ . A horocycle in  $P$  is a circle contained in  $P$  which touches  $\partial P$  at one point (see Fig. 4). Then the horocycle flow  $T_1P \rightarrow T_1P$  is the flow moving an unit tangent vector along a horocycle (in positive direction at unit speed). As above the polygon  $P$  determines a surface  $S$  of genus  $g > 1$  with abelian torsion-less fundamental group  $\pi_1(S)$  so that the homomorphism  $\pi_1(S) \rightarrow \mathbb{R}$  determines an unique (ergodic invariant) Radon measure. Finally the horocycle flow determines a factor  $II_\infty$  foliation associated to the factor  $III_1$  foliation. We remark for later usage that this foliation is determined by a set of closed curves (the horocycles). Using results of previous papers and subsections above, we have the following picture:

1. Every small exotic  $\mathbb{R}^4$  is determined by a codimension-1 foliation (unique up to cobordisms) of some homology 3-sphere  $\Sigma$  (as boundary  $\partial A = \Sigma$  of a contractable submanifold  $A \subset \mathbb{R}^4$ , the Akbulut cork). (see Theorem 5)
2. This codimension-1 foliation on  $\Sigma$  determines via surgery along a link uniquely a codimension-1 foliation on the 3-sphere and vice verse. (see Theorem 1)
3. This codimension-1 foliation  $(S^3, F)$  on  $S^3$  has a leaf space which is determined by the von Neumann algebra  $W(S^3, F)$  associated to the foliation. (see Connes [23])

<sup>7</sup> The group  $\mathbb{R}_+^*$  is the group of positive real numbers with multiplication as group operation also known as Pontrjagin dual.



**Figure 4** horocycle, a curve whose normals all converge asymptotically

4. The von Neumann algebra  $W(S^3, F)$  contains a hyperfinite factor  $III_1$  algebra as well as a factor  $I_\infty$  algebra coming from the Reeb foliations. (see Hurder and Katok [39])

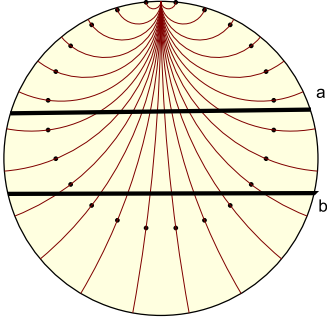
Thus by this procedure we get a noncommutative algebra from an exotic  $\mathbb{R}^4$ . The relation to the quantum theory will be discussed now. We remark that we have already a quantum theory represented by the von Neumann algebra  $W(S^3, F)$ . Thus we are in the strange situation to construct a (classical) Poisson algebra together with a quantization to get an algebra which we already have.

#### 4 The connection between exotic smoothness and quantization

In this section we describe a deep relation between quantization and the codimension-1 foliation of the  $S^3$  determining the smoothness structure on a small exotic  $\mathbb{R}^4$ . Here and in the following *we will identify the leaf space with its operator algebra*.

##### 4.1 Idempotent operators, closed curves in surfaces and knot cobordisms

In subsection 2.4, an idempotent was constructed in the  $C^*$  algebra of the foliation and geometrically interpreted as closed curve transversal to the foliation. Such a curve meets every leaf in a finite number of points. Furthermore the foliation on the 3-sphere  $S^3$  is determined up to foliated cobordisms (see Theorem 5), i.e. a 4-space which looks like  $S^3 \times [0, 1]$ . Then we have a cobordism of two curves which looks like a thickened curve  $S^1 \times [0, 1]$ . The foliation of the 3-sphere is determined by a polygon  $P$  (see subsection 2.2) laminated by curves starting and ending at the boundary of  $P$  (see Fig. 4). Without loss of generality we can assume that  $P$  consists of an even number of vertices, say  $2k$ . By the uniformization theorem of surfaces, there is a unique surface  $S$  of genus  $g > 1$  with  $g = [k/2]$  (with one boundary component for  $k$  odd) represented by  $P$ . The closed curves at  $S$  transversal to the foliation are represented by lines perpendicular to the leaves in the foliation of  $P$  (see Fig. 5). In the process from  $P$  to the surface  $S$  (via the identification of sides of  $P$ ) these lines in  $P$  close to curves in  $S$ . How do these curves look like? Usually a closed curve in the foliation of the 3-sphere is given by an



**Figure 5**  $a, b$  are examples of closed curves in  $P$  where the polygon is simplified by a circle

embedding  $S^1 \rightarrow S^3$  where the normal of this curve is in the direction of the leaf. This embedding is also known as a knot. In the process from  $P$  to  $S$ , we project this knot to the surface and get a closed curve with singularities (instead of crossings). Then this closed curve is represented by a line (in  $P$ ) perpendicular to the flow lines of the foliation. Now we will state the following theorem:

**Theorem 7** *Given a codimension-1 foliation of the 3-sphere represented by a polygon  $P$  with  $4k$  vertices. The corresponding Anosov foliation of the unit tangent bundle  $T_1S$  of a surface  $S$  with genus  $g = k$  admits idempotent operators in the leaf space given by closed curves in  $S$  with self-intersections. Two closed curves are equivalent if there is an isotopy between both curves.*

*Proof* The proof is a simple combination of results presented above. By definition (see 2.4), one has an idempotent operator in the leaf space of the foliation which is a closed curve transversal to the foliation. These curves are an embedding  $S^1 \rightarrow S^3$  known as knot. A deformation between two curves is an isotopy. Then both curves are equivalent.  $\square$

Thus we have to consider closed curves in surfaces.

#### 4.2 The observable algebra and Poisson structure

In this section we will describe the formal structure of a classical theory coming from the algebra of observables using the concept of a Poisson algebra. In quantum theory, an observable is represented by a hermitean operator having the spectral decomposition via projectors or idempotent operators. The coefficient of the projector is the eigenvalue of the observable or one possible result of a measurement. At least one of these projectors represent (via the GNS representation) a quasi-classical state. Thus to construct the substitute of a classical observable algebra with Poisson algebra structure we have to concentrate on the idempotents in the  $C^*$  algebra. Now we will see that the set of closed curves on a surface has the structure of a Poisson algebra. Let us start with the definition of a Poisson algebra.

**Definition 2** Let  $P$  be a commutative algebra with unit over  $\mathbb{R}$  or  $\mathbb{C}$ . A *Poisson bracket* on  $P$  is a bilinearform  $\{, \} : P \otimes P \rightarrow P$  fulfilling the following 3 conditions:

$$\text{anti-symmetry } \{a, b\} = -\{b, a\}$$

$$\text{Jacobi identity } \{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$$

derivation  $\{ab, c\} = a\{b, c\} + b\{a, c\}$ .

Then a *Poisson algebra* is the algebra  $(P, \{, \})$ .

Now we consider a surface  $S$  together with a closed curve  $\gamma$ . Additionally we have a Lie group  $G$  given by the isometry group. The closed curve is one element of the fundamental group  $\pi_1(S)$ . From the theory of surfaces we know that  $\pi_1(S)$  is a free abelian group. Denote by  $Z$  the free  $\mathbb{K}$ -module ( $\mathbb{K}$  a ring with unit) with the basis  $\pi_1(S)$ , i.e.  $Z$  is a freely generated  $\mathbb{K}$ -modul. Recall Goldman's definition of the Lie bracket in  $Z$  (see [32]). For a loop  $\gamma : S^1 \rightarrow S$  we denote its class in  $\pi_1(S)$  by  $\langle \gamma \rangle$ . Let  $\alpha, \beta$  be two loops on  $S$  lying in general position. Denote the (finite) set  $\alpha(S^1) \cap \beta(S^1)$  by  $\alpha \# \beta$ . For  $q \in \alpha \# \beta$  denote by  $\epsilon(q; \alpha, \beta) = \pm 1$  the intersection index of  $\alpha$  and  $\beta$  in  $q$ . Denote by  $\alpha_q \beta_q$  the product of the loops  $\alpha, \beta$  based in  $q$ . Up to homotopy the loop  $(\alpha_q \beta_q)(S^1)$  is obtained from  $\alpha(S^1) \cup \beta(S^1)$  by the orientation preserving smoothing of the crossing in the point  $q$ . Set

$$[\langle \alpha \rangle, \langle \beta \rangle] = \sum_{q \in \alpha \# \beta} \epsilon(q; \alpha, \beta) (\alpha_q \beta_q) \quad . \quad (7)$$

According to Goldman [32], Theorem 5.2, the bilinear pairing  $[\cdot, \cdot] : Z \times Z \rightarrow Z$  given by (7) on the generators is well defined and makes  $Z$  to a Lie algebra. The algebra  $Sym(Z)$  of symmetric tensors is then a Poisson algebra (see Turaev [60]).

The whole approach seems natural for the construction of the Lie algebra  $Z$  but the introduction of the Poisson structure is an artificial act. From the physical point of view, the Poisson structure is not the essential part of classical mechanics. More important is the algebra of observables, i.e. functions over the configuration space forming the Poisson algebra. Thus we will look for the algebra of observables in our case. For that purpose, we will look at geometries over the surface. By the uniformization theorem of surfaces, there is three types of geometrical models: spherical  $S^2$ , Euclidean  $\mathbb{E}^2$  and hyperbolic  $\mathbb{H}^2$ . Let  $\mathcal{M}$  be one of these models having the isometry group  $Isom(\mathcal{M})$ . Consider a subgroup  $H \subset Isom(\mathcal{M})$  of the isometry group acting freely on the model  $\mathcal{M}$  forming the factor space  $\mathcal{M}/H$ . Then one obtains the usual (closed) surfaces  $S^2$ ,  $\mathbb{R}P^2$ ,  $T^2$  and its connected sums like the surface of genus  $g$  ( $g > 1$ ). For the following construction we need a group  $G$  containing the isometry groups of the three models. Furthermore the surface  $S$  is part of a 3-manifold and for later use we have to demand that  $G$  has to be also a isometry group of 3-manifolds. According to Thurston [58] there are 8 geometric models in dimension 3 and the largest isometry group is the hyperbolic group  $PSL(2, \mathbb{C})$  isomorphic to the Lorentz group  $SO(3, 1)$ . It is known that every representation of  $PSL(2, \mathbb{C})$  can be lifted to the spin group  $SL(2, \mathbb{C})$ . Thus the group  $G$  fulfilling all conditions is identified with  $SL(2, \mathbb{C})$ . This choice fits very well with the 4-dimensional picture.

Now we introduce a principal  $G$  bundle on  $S$ , representing a geometry on the surface. This bundle is induced from a  $G$  bundle over  $S \times [0, 1]$  having always a flat connection. Alternatively one can consider a homomorphism  $\pi_1(S) \rightarrow G$  represented as holonomy functional

$$hol(\omega, \gamma) = \mathcal{P} \exp \left( \int_{\gamma} \omega \right) \in G \quad (8)$$

with the path ordering operator  $\mathcal{P}$  and  $\omega$  as flat connection (i.e. inducing a flat curvature  $\Omega = d\omega + \omega \wedge \omega = 0$ ). This functional is unique up to conjugation induced by a gauge

transformation of the connection. Thus we have to consider the conjugation classes of maps

$$hol : \pi_1(S) \rightarrow G$$

forming the space  $X(S, G)$  of gauge-invariant flat connections of principal  $G$  bundles over  $S$ . Now (see [47]) we can start with the construction of the Poisson structure on  $X(S, G)$ . The construction based on the Cartan form as the unique bilinearform of a Lie algebra. As discussed above we will use the Lie group  $G = SL(2, \mathbb{C})$  but the whole procedure works for every other group too. Now we consider the standard basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra  $sl(2, \mathbb{C})$  with  $[X, Y] = H$ ,  $[H, X] = 2X$ ,  $[H, Y] = -2Y$ . Furthermore there is the bilinearform  $B : sl_2 \otimes sl_2 \rightarrow \mathbb{C}$  written in the standard basis as

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Now we consider the holomorphic function  $f : SL(2, \mathbb{C}) \rightarrow \mathbb{C}$  and define the gradient  $\delta_f(A)$  along  $f$  at the point  $A$  as  $\delta_f(A) = Z$  with  $B(Z, W) = df_A(W)$  and

$$df_A(W) = \left. \frac{d}{dt} f(A \cdot \exp(tW)) \right|_{t=0} .$$

The calculation of the gradient  $\delta_{tr}$  for the trace  $tr$  along a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given by

$$\delta_{tr}(A) = -a_{21}Y - a_{12}X - \frac{1}{2}(a_{11} - a_{22})H \quad .$$

Given a representation  $\rho \in X(S, SL(2, \mathbb{C}))$  of the fundamental group and an invariant function  $f : SL(2, \mathbb{C}) \rightarrow \mathbb{R}$  extendable to  $X(S, SL(2, \mathbb{C}))$ . Then we consider two conjugacy classes  $\gamma, \eta \in \pi_1(S)$  represented by two transversal intersecting loops  $P, Q$  and define the function  $f_\gamma : X(S, SL(2, \mathbb{C})) \rightarrow \mathbb{C}$  by  $f_\gamma(\rho) = f(\rho(\gamma))$ . Let  $x \in P \cap Q$  be the intersection point of the loops  $P, Q$  and  $c_x$  a path between the point  $x$  and the fixed base point in  $\pi_1(S)$ . Then we define  $\gamma_x = c_x \gamma c_x^{-1}$  and  $\eta_x = c_x \eta c_x^{-1}$ . Finally we get the Poisson bracket

$$\{f_\gamma, f'_\eta\} = \sum_{x \in P \cap Q} \text{sign}(x) B(\delta_f(\rho(\gamma_x)), \delta_{f'}(\rho(\eta_x))) \quad ,$$

where  $\text{sign}(x)$  is the sign of the intersection point  $x$ . Thus,

**Theorem 8** *The space  $X(S, SL(2, \mathbb{C}))$  has a natural Poisson structure (induced by the bilinear form (7) on the group) and the Poisson algebra  $(X(S, SL(2, \mathbb{C})), \{, \})$  of complex functions over them is the algebra of observables.*

### 4.3 Drinfeld-Turaev Quantization

Now we introduce the ring  $\mathbb{C}[[h]]$  of formal polynomials in  $h$  with values in  $\mathbb{C}$ . This ring has a topological structure, i.e. for a given power series  $a \in \mathbb{C}[[h]]$  the set  $a + h^n \mathbb{C}[[h]]$  forms a neighborhood. Now we define

**Definition 3** A *Quantization* of a Poisson algebra  $P$  is a  $\mathbb{C}[[h]]$  algebra  $P_h$  together with the  $\mathbb{C}$ -algebra isomorphism  $\Theta : P_h/hP \rightarrow P$  so that

1. the modul  $P_h$  is isomorphic to  $V[[h]]$  for a  $\mathbb{C}$  vector space  $V$
2. let  $a, b \in P$  and  $a', b' \in P_h$  be  $\Theta(a) = a', \Theta(b) = b'$  then

$$\Theta\left(\frac{a'b' - b'a'}{h}\right) = \{a, b\}$$

One speaks of a deformation of the Poisson algebra by using a deformation parameter  $h$  to get a relation between the Poisson bracket and the commutator. Therefore we have the problem to find the deformation of the Poisson algebra  $(X(S, SL(2, \mathbb{C})), \{, \})$ . The solution to this problem can be found via two steps:

1. at first find another description of the Poisson algebra by a structure with one parameter at a special value and
2. secondly vary this parameter to get the deformation.

Fortunately both problems were already solved (see [59,60]). The solution of the first problem is expressed in the theorem:

**Theorem 9** *The Skein modul  $K_{-1}(S \times [0, 1])$  (i.e.  $t = -1$ ) has the structure of an algebra isomorphic to the Poisson algebra  $(X(S, SL(2, \mathbb{C})), \{, \})$ . (see also [20, 19])*

Then we have also the solution of the second problem:

**Theorem 10** *The skein algebra  $K_t(S \times [0, 1])$  is the quantization of the Poisson algebra  $(X(S, SL(2, \mathbb{C})), \{, \})$  with the deformation parameter  $t = \exp(h/4)$ . (see also [20])*

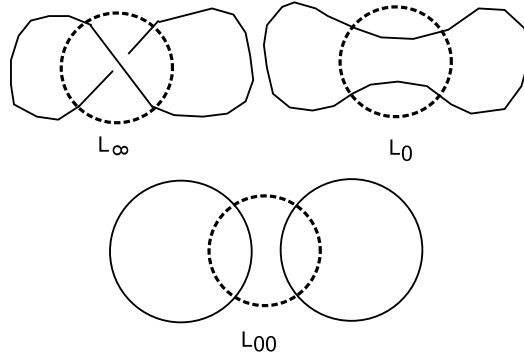
To understand these solutions we have to introduce the skein module  $K_t(M)$  of a 3-manifold  $M$  (see [46]). For that purpose we consider the set of links  $\mathcal{L}(M)$  in  $M$  up to isotopy and construct the vector space  $\mathbb{C}\mathcal{L}(M)$  with basis  $\mathcal{L}(M)$ . Then one can define  $\mathbb{C}\mathcal{L}[[t]]$  as ring of formal polynomials having coefficients in  $\mathbb{C}\mathcal{L}(M)$ . Now we consider the link diagram of a link, i.e. the projection of the link to the  $\mathbb{R}^2$  having the crossings in mind. Choosing a disk in  $\mathbb{R}^2$  so that one crossing is inside this disk. If the three links differ by the three crossings  $L_{oo}, L_o, L_\infty$  (see figure 6) inside of the disk then these links are skein related. Then in  $\mathbb{C}\mathcal{L}[[t]]$  one writes the skein relation<sup>8</sup>  $L_\infty - tL_o - t^{-1}L_{oo}$ . Furthermore let  $L \sqcup O$  be the disjoint union of the link with a circle then one writes the framing relation  $L \sqcup O + (t^2 + t^{-2})L$ . Let  $S(M)$  be the smallest submodule of  $\mathbb{C}\mathcal{L}[[t]]$  containing both relations, then we define the Kauffman bracket skein modul by  $K_t(M) = \mathbb{C}\mathcal{L}[[t]]/S(M)$ . We list the following general results about this modul:

- The modul  $K_{-1}(M)$  for  $t = -1$  is a commutative algebra.
- Let  $S$  be a surface then  $K_t(S \times [0, 1])$  carries the structure of an algebra.

---

<sup>8</sup> The relation depends on the group  $SL(2, \mathbb{C})$ .





**Figure 6** crossings  $L_\infty, L_0, L_{00}$

The algebra structure of  $K_t(S \times [0, 1])$  can be simple seen by using the diffeomorphism between the sum  $S \times [0, 1] \cup_S S \times [0, 1]$  along  $S$  and  $S \times [0, 1]$ . Then the product  $ab$  of two elements  $a, b \in K_t(S \times [0, 1])$  is a link in  $S \times [0, 1] \cup_S S \times [0, 1]$  corresponding to a link in  $S \times [0, 1]$  via the diffeomorphism. The algebra  $K_t(S \times [0, 1])$  is in general non-commutative for  $t \neq -1$ . For the following we will omit the interval  $[0, 1]$  and denote the skein algebra by  $K_t(S)$ . Furthermore we remark, that *all results remain true if we use an intersection in  $L_\infty$  instead of a crossing.*

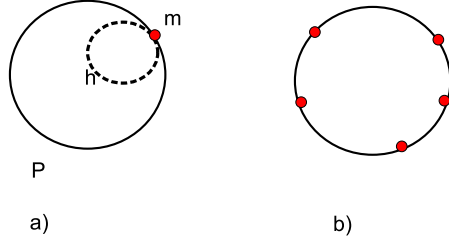
*Ad hoc* the skein algebra is not directly related to the foliation. We used only the fact that there is an idempotent in the  $C^*$  algebra represented by a closed curve. It is more satisfying to obtain a direct relation between both construction. Then the von Neumann algebra of the foliation is the result of a quantization in the physical sense. This construction is left for the next subsection.

#### 4.4 Temperley-Lieb algebra and the operator algebra of the foliation

In this subsection we will describe a direct relation between the skein algebra and the factor  $III_1$  constructed above. At first we will summarize some of the results above.

1. The foliation of the 3-sphere  $S^3$  has non-trivial Godbillon-Vey class. The corresponding von Neumann algebra must contain a factor  $III$  algebra.
2. We obtained that the von Neumann algebra is the hyperfinite factor  $III_1$  determined by a factor  $II_\infty$  algebra via Tomita-Takesaki theory.
3. In the von Neumann algebra there are idempotent operators given by closed curves in the foliation.
4. The set of closed curves carries the structure of the Poisson algebra whose quantization is the skein algebra determined by knots and links. Thus the skein algebra can be seen as a quantization of the fundamental group.

Thus our main goal in this subsection should be a direct relation between a suitable skein algebra and the von Neumann algebra of the foliation. As a first step we remark that a factor  $II_\infty$  algebra is the tensor product  $II_\infty = II_1 \otimes I_\infty$ . Thus the main factor is given by the  $II_1$  factor, i.e. a von Neumann algebra with finite trace. From the point of view of invariants, both factors  $II_\infty$  and  $II_1$  are Morita-equivalent leading to the same K-theoretic invariants.



**Figure 7** fig. a) an example for a horocycle  $h$  on  $P$  and fig. b) marked points on the boundary of the polygon  $P$

Now we are faced with the question: Is there any skein algebra isomorphic to the factor  $II_1$  algebra? Usually the skein algebra is finite or finitely generated (as module over the first homology group). Thus we have to construct a finite algebra reconstructing the factor  $II_1$  in the limit. Following the theory of Jones [40], one uses a tower of Temperley-Lieb algebras as generated by projection (or idempotent) operators. Thus, if we are able to show that a skein algebra constructed from the foliation is isomorphic to the Temperley-Lieb algebra then we have constructed the factor  $II_1$  algebra.

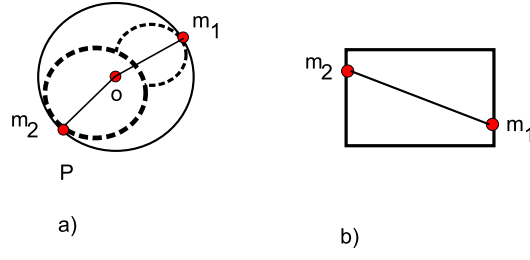
For the construction we go back to factor  $II_\infty$  foliation discussed above and identified as the horocycle foliation. Let  $P$  the polygon with hyperbolic metric used in subsection 2.5 and in subsection 2.2. Given a polygon  $P$  as covering space of a surface  $S$  (of genus  $g > 1$ ) with non-positive curvature. Denote by  $\gamma_v$  the geodesic with initial tangent vector  $v$  and by  $dist(\gamma_v(t), \gamma_w(t))$  the distance between two points on two curves. We call the two tangent vectors  $v, w$  of the cover  $P$  asymptotic if the distance  $dist(\gamma_v(t), \gamma_w(t))$  is bounded as  $t \rightarrow \infty$ . For a unit tangent vector  $v \in T_1P$  define the Busemann function  $b_v : P \rightarrow \mathbb{R}$  by

$$b_v(q) = \lim_{t \rightarrow \infty} (dist(\gamma_v(t), q) - t)$$

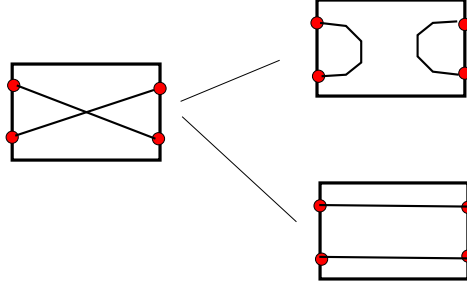
This function is differentiable and the gradient  $-\nabla_q b_v$  is the unique vector at  $q$  asymptotic to  $v$ . We define alternatively the horocycle  $h(v)$  (determined by  $v$ ) as the level set  $b_v^{-1}(0)$ . Clearly  $h(v)$  is the limit as  $R \rightarrow \infty$  of the geodesic circles of radius  $R$  centered at  $\gamma_v(R)$ . Let  $W(v)$  be the set of vectors  $w$  asymptotic to  $v$  with footpoints on  $h(v)$  (see Fig. 4), i.e.

$$W(v) = \{-\nabla_q b_v \mid q \in h(v)\}.$$

The curves  $W(v)$ ,  $v \in T_1P$  are the leaves of the horocycle foliation  $W$  of  $T_1P$  which can be lifted to a horocycle foliation  $W$  on  $T_1S$ . Remember a horocycle is a circle in the interior of  $P$  touching the boundary at one point. Now we consider the flow in  $T_1P$  along a horocycle with unit speed which induces a codimension-1 foliation in  $T_1P$ . The horocycle foliation is parametrized by the set of horocycles on  $P$ . Thus the set of unit tangent vectors labels the leaves of the foliation or the leaf space is parametrized by unit tangent vectors. Furthermore we remark that every horocycle is also determined by a unit tangent vector. By definition, the set of unit tangent vectors is completely determined by curves in  $P$ . Every horocycle meets the boundary of  $P$  at one point, which we mark (see Fig. 7), say  $m_1, \dots, m_n$ . Then by uniqueness of the flow, there is a curve from the boundary point  $m_1$  in the interior of  $P$  meeting a point  $o$  followed by



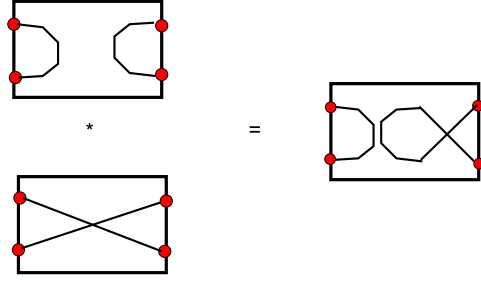
**Figure 8** a) flow line between two marked points defined via two horocycles, b) simple picture as substitute



**Figure 9** resolution of the flow singularities

a curve from this point  $o$  to another boundary point  $m_2$  (see figure 8). Thus in general we obtain curves in  $P$  going from one marked boundary point  $m_k$  to another marked boundary point  $m_l$ . We need only a countable number of these points. Therefore we choose the number of vertices  $k$  of the polygon  $P$ . Without loss of generality we choose an even number of vertices  $k = 2n$ . Then any pair of vertices is connected by one geodesic path. To express the grouping of the marked points, we used a rectangle instead of the circle (as indicated in Fig. 8). All other paths can be generated by a simple variation of the start and end point (isotopy). Using the horocycles we obtain a flow for every pair of marked points. The set of unit tangent vectors labels the leaves of the foliation and can be described by curves between the marked points. Then we group the marked points and assume that we have the same number of marked points on the left and on the right side of  $P$ . Now we have to define the (formal) sum of two flows. A flow starts on one marked point of one side going to one point at the other side (see figure 8). By using this definition we obtain also singularities, i.e. crossings of flows. But the singularities or intersection points can be solved to get non-singular flows. The figure 9 shows the method<sup>9</sup>. In the subsection 4.3 we introduced the skein algebra. We define the resolution of the singularity together with a parameter  $t$  in similarity to the Skein relation  $L_\infty = tL_o + t^{-1}L_{oo}$ . By this method we are also able to define a sum of two flows by reversing the procedure: the sum of two flows will produce a singular flow. Now we consider two polygons with the same number of marked points on one side. These polygons can be put together (see figure 10) to define a product. Before

<sup>9</sup> The method was used in the theory of finite knot invariants (Vassiliev invariants) and is known as STU relation.



**Figure 10** product structure

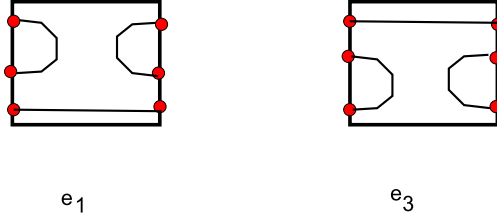
presenting the theorem, we summarize the definition of the operations:

1. Given a polygon  $P(k, k)$  with  $2k$  marked points, i.e. with  $k$  marked points of each side. There is only one internal line between two marked points, i.e.  $2k$  marked points are connected by  $k$  lines. Two internal lines do not intersect. The marked points of each side have equal distances to each other. One can add or remove an internal circle.
2. *Product*: The connected sum of polygons  $P(k, k)$  with  $k$  marked points on each side is the product in the algebra (see Fig. 10).
3. *Multiplication by number*: The change in the distance between two marked points is the multiplication with a real number. The details of this operation is not important at the moment.
4. *Sum*: A linear combination between two polygons is represented by the crossing of two internal lines (see Fig. 9).
5. *\*operation*: A  $180^\circ$  rotation of the polygon  $P^*(k, k)$  is the \*operation.

Putting all these definitions together we obtain:

**Theorem 11** *The leaf space of the horocycle foliation of a surface  $S$  (of genus  $g > 1$ ) is represented by the leaf space of a horocycle foliation of a polygon  $P$  in  $\mathbb{H}^2$ . If one extends the leaf space to allow (countable many) crossings between two leaves and considers a connected sum of the polygons then this extended leaf space admits the structure of an  $*$ -algebra (see the rules above). Let  $P(k, k)$  be a polygon with  $2k$  vertices generating a foliation (see subsection 2.2) with  $*$ -algebra  $P_k$  as extended leaf space. This algebra is isomorphic to the Temperley-Lieb-algebra  $TL_k$  generated by  $k$  elements  $\{e_1, \dots, e_n\}$  subject to the relations (9). The generators are idempotent operators represented by closed curves in  $S$ . In the direct limit one obtains the hyperfinite factor  $II_1$  algebra given as tower of Temperley-Lieb algebras (and equal to the skein algebra of a marked disk).*

*Proof* The relation between the two foliations was already shown above. The extended leaf space was also constructed above. Especially we mention the operations. The polygon  $P(k, k)$ , where each internal line connects one marked point on one side with a marked point on the other side, is the identity in the algebra. The next complicated case is given by a polygon where two marked points on each side are connected by one internal line. We denote these polygons by  $e_n$  and refer to figure 11 for the convention. The product operation is defined above. By simple graphical manipulations (see [46]



**Figure 11** example for generators  $e_1$  and  $e_3$  of the Temperley-Lieb algebra

§26.9.) using our rules above, we obtain the relations:

$$\begin{aligned} e_i^2 &= e_i, \quad e_i e_j = e_j e_i : |i - j| > 1, \\ e_i e_{i+1} e_i &= e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad e_i^* = e_i \end{aligned} \quad (9)$$

where the  $*$ operation obviously do not change the generators  $e_i$ . This algebra is the Temperley-Lieb algebra  $TL_n$  generated by  $\{e_1, \dots, e_n\}$ . The inclusion  $TL_n \subset TL_{n+1}$  is given by adding two marked points, one point on each side connected by one internal line. As Jones [40] showed: the limit case  $\lim_{n \rightarrow \infty} TL_n$  (considered as direct limit) is the factor  $II_1$ . Thus we have constructed the factor  $II_1$  algebra as skein algebra.  $\square$

We will finish this subsection with one remark. In the factor Temperley-Lieb algebra there is an unique idempotent operator, the Jones-Wenzl idempotent, which is related to Connes idempotent operator in the operator algebra of the foliation by our construction.

## 5 Knots as states

Lets summarize the situation again:

1. A small exotic  $\mathbb{R}^4$  is related to a codimension-1 foliation of the 3-sphere  $S^3$  unique up to foliated cobordism.
2. The leaf space of this foliation is a factor  $III_1$  algebra related to a foliation with leaf space a factor  $II_\infty = II_1 \otimes I_\infty$  algebra.
3. The algebra  $II_1$  is the direct limit of the Temperley-Lieb algebras  $TL_n$ .
4. Closed curves transversal to the foliation are related to idempotent operators of the leaf space.

This section is two-fold. On one side we will try to construct the states in the factor  $III_1$  algebra by geometric methods. But on the other side we will also discuss the operators interpreted as cobordisms between the states (in the sense of the topological QFT a la Atiyah). This section is different from the others, we present only the ideas and leave the proofs for further papers.

### 5.1 Observables as closed curves

In subsection 4.2 we described the observable algebra (i.e. the Poisson algebra) as the space of flat connections  $X(S, SL(2, \mathbb{C}))$  where the observables are the holonomies (8)

of the flat connection along closed curves. This picture remains true after quantization where we obtain the skein algebra. In subsection 4.1 we considered the idempotent operators represented by closed curves. Finally we showed in Theorem 11 that these closed curves generate the Temperley-Lieb algebra. Especially the linear combination of two generators is represented by a singular flow (see Fig. 9). An observable is represented by a hermitean operator or equivalently by a linear combination of projectors. Therefore an observable is given by a singular flow or by using Theorem 7 as a singular knot (or link).

Now we will consider a special idempotent operator known as Jones-Wenzl idempotent. For its construction we have to modify the definition (9) of the algebra: instead of  $e_i^2 = e_i$  we write

$$e_i^2 = \tau e_i$$

where  $\tau$  is a real number given by a closed circle in the polygon  $P(k, k)$ . So, we modify one rule in the definition of our algebra: adding an internal circle is equivalent to a multiplication with  $\tau$ . If  $\tau$  is the number  $\tau = a_0^2 + a_0^{-2}$  with  $a_0$  a  $4n$ th root of unity ( $a_0^{4k} \neq 1$  for  $k = 1, \dots, n-1$ ) and  $A_n \subset TL_n$  a subalgebra generated of  $\{e_1, \dots, e_n\}$  missing the identity  $1_n$  then there is an element  $f^{(n)}$  with

$$\begin{aligned} f^{(n)} A_n &= A_n f^{(n)} = 0 \\ 1_n - f^{(n)} &\in A_n \\ f^{(n)} f^{(n)} &= f^{(n)} \end{aligned}$$

called the Jones-Wenzl idempotent [46]. This idempotent is used to define a 3-manifold invariant of Witten type [62]. Witten defines this invariant by the state sum of the Chern-Simons theory (for a suitable gauge group, here it is  $SU(2)$ ). This unexpected relation gives a hint for a possible action related to our QFT (given by the factor  $III_1$ ).

## 5.2 Knot concordance and capped gropes

In the previous subsection we considered the observables of the theory given by singular knots or links. Now we are interested in the construction of states in the factor  $II_1$ -algebra  $A$ , i.e. linear functionals  $f : A \rightarrow \mathbb{C}$  which are positive ( $f(x^*x) \geq 0$  for all  $x \in A$ ) and normed  $\|f\| = 1$ . Usually one constructs the states by a representation (GNS) of the algebra  $A$  into a Hilbert space. From the physical point of view, one can argue that every pure state must be corresponding to a classical state. Because of the relation between the Poisson algebra  $X(S, SL(2, \mathbb{C}))$  and its quantization as skein space  $K_t(S \times [0, 1])$ , the state must be the holonomy along a knot. To see this fact, we will follow another path. Our theory is purely topological, i.e. we assign to every observable (as endomorphism of some Hilbert space) a singular knot. Following the axioms of a Topological QFT (TQFT) by Atiyah, one assigns to a cylinder like  $S \times [0, 1]$  an endomorphism. In our case it is an element of the skein space, i.e. a singular knot. Singular knots appear in the theory of Vassiliev invariants where one considers transitions between two different knots. For our approach we have to interpret the 3-dimensional objects in the 4-dimensional context.

For that purpose we consider the horocycle foliation of the unit tangent bundle  $T_1 S$  of a surface  $S$  with genus  $g > 1$ . As usual we associate to this foliation a codimension-1 foliation of the 3-sphere given by a polygon  $P$  in  $\mathbb{H}^2$  with  $4g$  vertices (or sides). In the

proof of the Theorem 5, the cobordism class of this foliation (i.e. the Godbillon-Vey invariant) is determined by a capped grope (with its design). This capped grope is determined by a path in a binary tree (a trivalent tree), see subsection 3.4. The capped grope is embedded in some 4-space, i.e. the boundary of the capped grope, a circle, is embedded in the 3-space ( $S^3$  or  $\mathbb{R}^3$ ). The caps in the capped grope are also immersed disks (see subsection 3.5). Therefore, by fixing one cap, we have a cobordism between two embedded circles or a cobordism between two knots. Especially, this cobordism is induced from a cobordism between the embedding spaces, usually called a knot concordance. Therefore we have identified the endomorphisms as knot concordance between two knots, the states. There is an extensive literature for the relation between knot concordances and capped gropes [54, 22]. Especially we mention the relation to Vassiliev invariants [55] and the Kontsevich integral [56]. Here we presented only this relation and refer to our future work.

## 6 Discussion

In this paper we presented a variety of relations between codimension-1 foliations of the 3-sphere  $S^3$  and noncommutative algebras. By using the results of our previous paper [9], we obtain a relation between (small) exotic smoothness of the  $\mathbb{R}^4$  and noncommutativity via the noncommutative leaf space of the foliation and the Casson handle. Thus we get our main result of this paper:

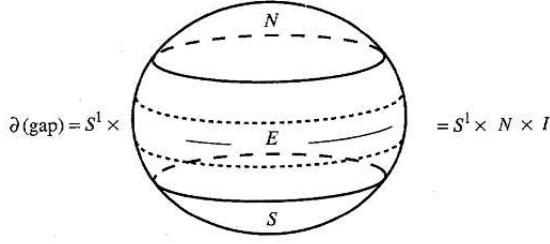
*The Casson handle carries the structure of a noncommutative space determined by a factor  $II_1$  algebra which is related to the skein algebra of the disk with marked points and to the leaf space of the horocycle foliation.*

Thus we have obtained a direct link between noncommutative spaces and exotic 4-manifolds which can be used to get a direct relation to quantum field theory. One of the central elements in the algebraic quantum field theory is the Tomita-Takesaki theory leading to the  $III_1$  factor as vacuum sector [14]. As a possible candidate one has loop quantum gravity with an unique diffeomorphism-invariant vacuum state [43]. Especially the relation to skein spaces and knot concordances are very attractive for future work. Our work has also some overlap with the nice work [10] for quantum gravity. We will close our paper with some speculations for a possible interpretation of the capped gropes as the trees in Connes-Kreimer renormalization theory. Starting point is the observation, that the Hopf algebra of formal vector fields in a codimension-1 foliation is isomorphic to the Hopf algebra of renormalization in QFT a la Kreimer. In our context it means that exotic smoothness (as described by codimension-1 foliations) has many to do with renormalization in QFT. Again we refer to our future work.

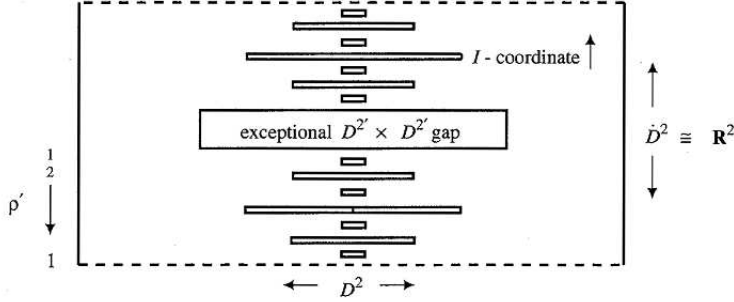
## Appendix

### A Proof of Theorem 5

*Proof* We consider a tubular neighborhood  $\partial A \times [0, 1] \subset \mathbb{R}_1^4$  of  $\partial A$  and glue the Casson handle along some 2-handle. Now we will weaken the Casson handles by using capped gropes (see chapter 1-4 in [28]) denoted by GCH. These differ from Casson handles in that many surface stages are interspersed between the immersed disks of Casson's construction. The GCH are also indexed by rooted finitely branching objects. The growth rate of their stages was determined in [5] (Theorem A) to be at least exponential (more than  $2^n$ ). In the proof of Theorem 3.2



**Figure 12** decomposition of the boundary of a gap (fig. repainted from [27])



**Figure 13** visualization of foliation as coordinatization of the design (fig. repainted from [27])

in [27] the gaps in the design where used. The gaps in case of the Casson handle are not manifolds and look like  $S^1 \times D^3/Wh$ . In case of the capped grope one has “good” gaps of the form  $S^1 \times D^3$ . That is the reason why we switch to these objects now. Now we decompose the gap by  $\text{gap} = S^1 \times D^3 = S^1 \times D^2 \times I$  with the unit interval  $I = [0, 1] = D^1$ . The boundary is a decomposition  $\partial(\text{gap}) = (S^1 \times S^1 \times I) \cup (S^1 \times D^2 \times \{0, 1\})$  of the caps (north and south) and the equator region (see Fig. 12). The radius coordinate  $\rho$  defined above is identified with the unit interval of the gap (see the proof of Theorem 3.2). In the notation of [27], we think of each gap as  $\text{gap} = S^1 \times N \times I$  where  $N = D^2$  is the neighborhood around the north pole of the 2-sphere in Fig. 12. Using the reembedding theorems every GCH embeds in the open 2-handle and induces a foliation visualized in Fig. . As described in subsection 3.4 the simplest tree  $Tree_+$  belongs to the binary sequence .000... and is represented by  $t = 0$  and the radius  $r = 1/(1 - t) = +\infty$ . The foliation of the design is perpendicular to  $S^1 \times N$ , i.e.  $S^1 \times \{\text{longitudes}\}$  are the leaves. The intersection of the leaves with  $S^1 \times N$  produces a foliation of the disk  $N$ . This disk is given up to conformal automorphism by fixing the sphere  $S^2 \supset N$ , i.e. the disk is invariant w.r.t. the group  $PSL(2, \mathbb{R})$ . The boundary of  $N$  is given by geodesic curves. The  $PSL(2, \mathbb{R})$ -invariance induces a mapping of the disk  $N$  into the hyperbolic space  $\mathbb{H}^2$ , where every  $PSL(2, \mathbb{R})$  transformation is an isometry now. Then the foliated  $N$  is mapped to a foliated polygon  $P$  in  $\mathbb{H}^2$ , where the foliation is  $PSL(2, \mathbb{R})$ -invariant. From this point of view we interpret  $S^1 \times N$  as the unit tangent bundle of the polygon  $T_1 P$ . Then the volume of the polygon  $P$  is the volume of the disk  $N$ , i.e.  $\text{vol}(P) = \text{vol}(N)$  and we choose the number of vertices of  $P$  in a suitable manner by defining the geodesic arcs forming the boundary of  $N$ . As Fig. 12 indicated, the disk  $N$  is also part of the boundary  $\partial(\text{gap}) = S^1 \times S^2$  of the gap. Then the unit interval in the gap is directly related to the radius  $r$  of the 2-sphere  $S^2 \supset N$  and this radius determines the volume of the disk  $N$  (as part of the upper hemisphere of  $S^2$ , see Fig. 12). But then by using  $PSL(2, \mathbb{R})$ -invariance, we obtain the relation  $\text{vol}(P) = r^2$ .

The tubular neighborhood  $\partial A \times [0, 1] = \partial A \times I$  can be chosen in such a manner that the coordinate  $\rho$  agrees with the unit interval  $I$  of the neighborhood. In subsection 3.4 we described the foliation of the design as a foliated cobordism between two disks  $N, S$  given by the 2-sphere



$S^2$  (see Fig. 2). As described above, every disk ( $N$  or  $S$ ) is related to the polygon  $P$  (without loss of generality we use  $N$  and  $S$  with the same volume  $\text{vol}(N) = \text{vol}(S)$ ). Then the foliation of the design induces a foliation of the cobordism  $\partial A \times I$ . The space  $D^2 \times S^2$  (the 5-stage towers) is the leaf of the foliated cobordism transverse to the foliation on the boundary  $\partial A$ . Then the restriction on the boundary  $\partial A \times \{0, 1\}$  induces a foliation on  $\partial A$  determined by the volume  $\text{vol}(P)$  of the polygon. So, the radius  $r^2$  (proportional to the volume of  $\text{vol}(P)$ ) is a cobordism invariant of the foliation. By Theorem 1 we obtain a codimension-1 foliation of  $\partial A$  induced from a foliation of the  $S^3$ . As shown [57] (see also the book [52] chapter VIII for the details) this invariant agrees with the Godbillon-Vey invariant  $GV = r^2$ . Then two non-diffeomorphic small exotic  $\mathbb{R}^4$  (for  $s \neq t$ ) have different radial coordinates ( $\frac{1}{1-s} = r_s \neq r_t = \frac{1}{1-t}$ ) and therefore different Godbillon-Vey invariants  $r_s^2 \neq r_t^2$ . The corresponding foliations are non-cobordant to each other.  $\square$

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